MATH 637: Mathematical Techniques in Data Science
Neural networks I

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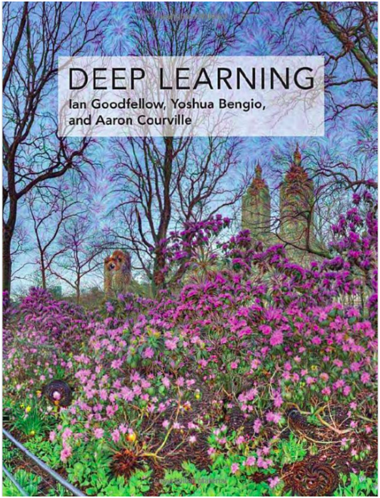
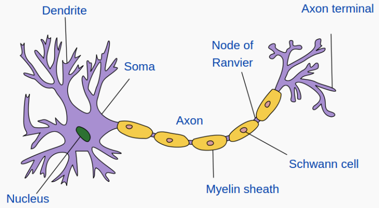
This lecture is based on the UFLDL tutorial (http://deeplearning.stanford.edu/tutorial/)

Reference



https://www.deeplearningbook.org/

Neurons



Neuron representation (Source: Wiki).

- Our brain contains about 86 billion neurons.

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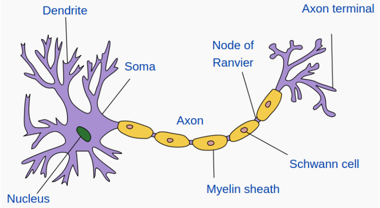

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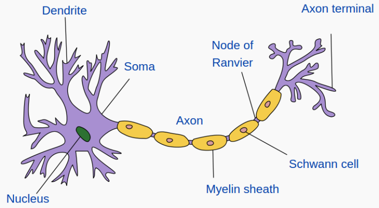

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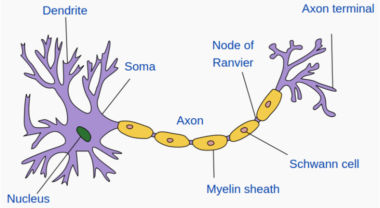

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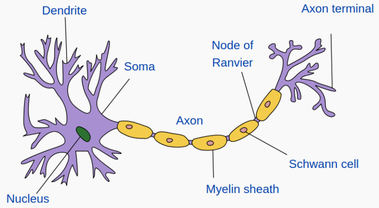

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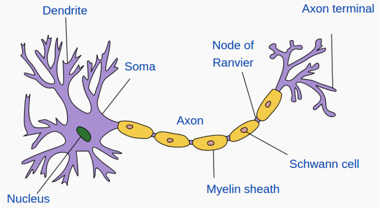

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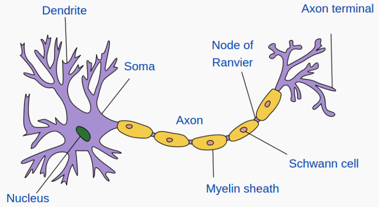

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- When a neuron fires, it starts a chain reaction that propagates information.
- There are *excitatory* and *inhibitory* synapses.

See Izenman (2013) for more details.

Neurons (cont.)

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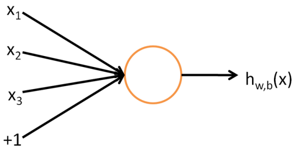
- Very popular in computer vision, natural language processing, and many other fields.
- Today, neural network models are often called *deep learning*.

Single neuron model:



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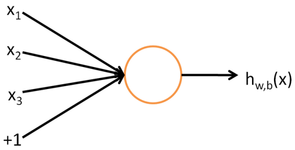

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Input: x_1, x_2, x_3 (and $+1$ intercept).

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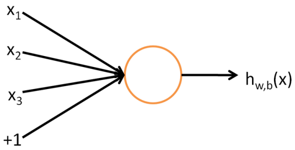

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Input: x_1, x_2, x_3 (and $+1$ intercept).
Output: $h_{W,b}(x) = f(W^T x) = f(W_1 x_1 + W_2 x_2 + W_3 x_3 + b)$,
where f is the *rectified linear unit (ReLU)* function:

$$f(x) = \max(0, x).$$

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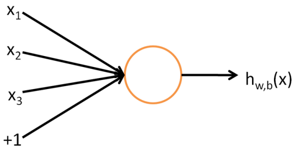

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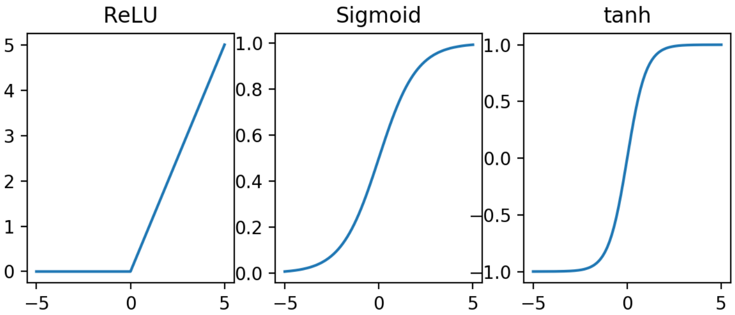
Other common choice for f are the sigmoid and the hyperbolic tangent:

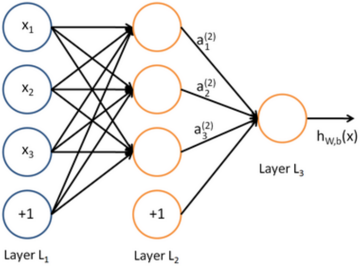
$$f(x) = \frac{1}{1+e^{-x}} \qquad f(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

The function f acts as an **activation** function.



Idea: Depending on the input of the neuron and the *strength* of the links, the neuron "fires" or not.

A **neural networks model** is obtained by hooking together many neurons so that the output of one neuron becomes the input of another neuron.



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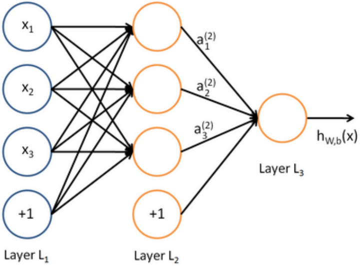

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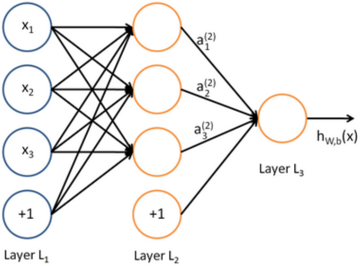

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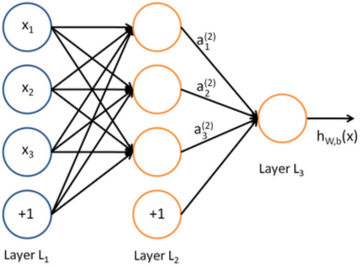

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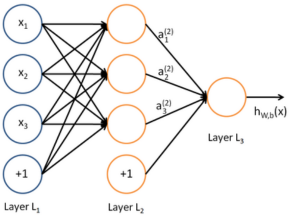
We will let n_l denote the **number of layers** in our model ($n_l = 3$ in the above example).

Notation



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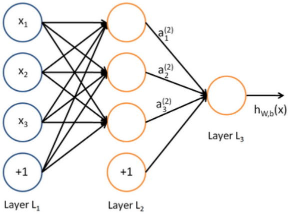

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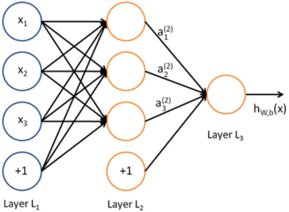

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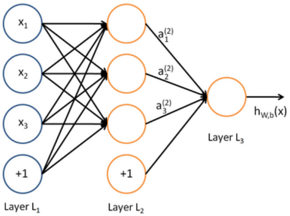

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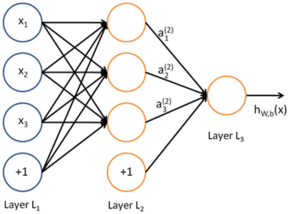

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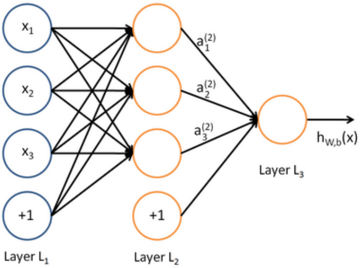
In above example: $(W, b) = (W^{(1)}, b^{(1)}, W^{(2)}, b^{(2)})$. Here $W^{(1)} \in \mathbb{R}^{3\times3}$, $W^{(2)} \in \mathbb{R}^{1\times3}$, $b^{(1)} \in \mathbb{R}^3$, $b^{(2)} \in \mathbb{R}$.

Activation



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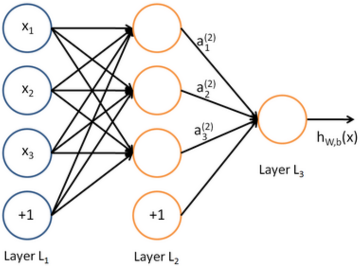

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We have:

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a_1^{(2)} &= f(W_{11}^{(1)}x_1 + W_{12}^{(1)}x_2 + W_{13}^{(1)}x_3 + b_1^{(1)}) \\
a_2^{(2)} &= f(W_{21}^{(1)}x_1 + W_{22}^{(1)}x_2 + W_{23}^{(1)}x_3 + b_2^{(1)}) \\
a_3^{(2)} &= f(W_{31}^{(1)}x_1 + W_{32}^{(1)}x_2 + W_{33}^{(1)}x_3 + b_3^{(1)}) \\
h_{W,b} = a_1^{(3)} &= f(W_{11}^{(2)}a_1^{(2)} + W_{12}^{(2)}a_2^{(2)} + W_{13}^{(2)}a_3^{(2)} + b_1^{(2)}).
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Compact notation

- In what follows, we will let $z_i^{(l)}$ = total weighted sum of inputs to unit i in layer l (including the bias term):

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We extend f *elementwise*: $f([v_1, v_2, v_3]) = [f(v_1), f(v_2), f(v_3)]$. Using the above notation, we have:

$$\begin{aligned} z^{(2)} &= W^{(1)} x + b^{(1)} \\ a^{(2)} &= f(z^{(2)}) \\ z^{(3)} &= W^{(2)} a^{(2)} + b^{(2)} \\ h_{W,b} &= a^{(3)} = f(z^{(3)}). \end{aligned}$$

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- In that case, we obtain a **feedforward neural network** (no directed loops or cycles).

Multiple outputs

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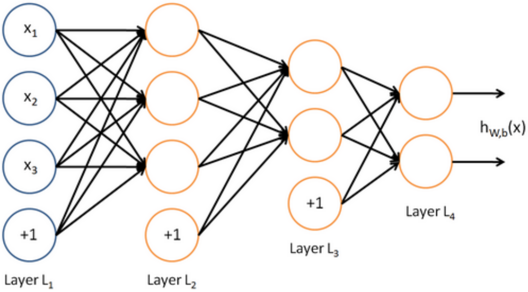

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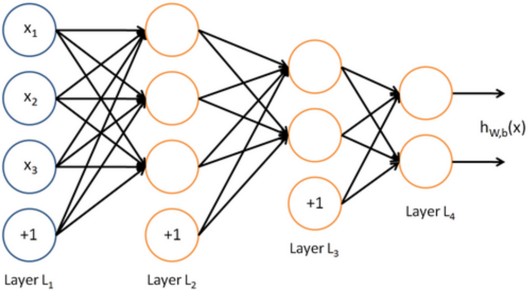

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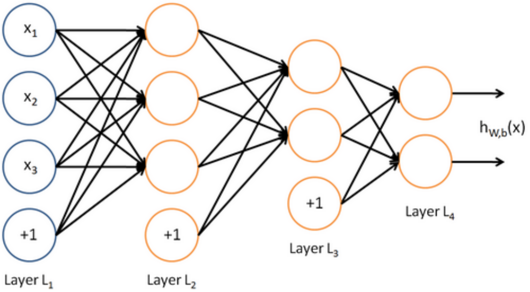

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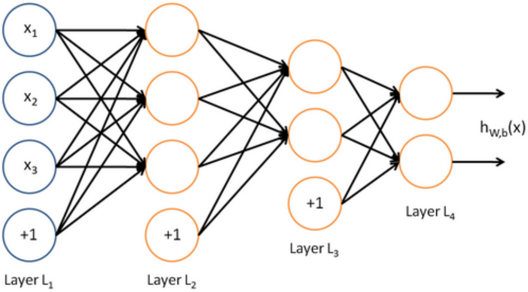

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- Useful to *encode* or *compress* information.