MATH 637: Mathematical Techniques in Data Science
Neural networks II

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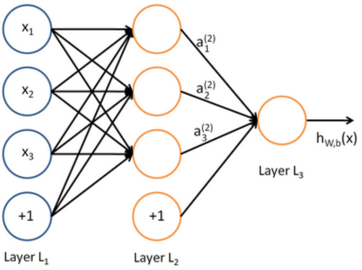
This lecture is based on the UFLDL tutorial (http://deeplearning.stanford.edu/tutorial/)

Recall



We have:

$$
\begin{aligned}
a_1^{(2)} &= f(W_{11}^{(1)}x_1 + W_{12}^{(1)}x_2 + W_{13}^{(1)}x_3 + b_1^{(1)}) \\
a_2^{(2)} &= f(W_{21}^{(1)}x_1 + W_{22}^{(1)}x_2 + W_{23}^{(1)}x_3 + b_2^{(1)}) \\
a_3^{(2)} &= f(W_{31}^{(1)}x_1 + W_{32}^{(1)}x_2 + W_{33}^{(1)}x_3 + b_3^{(1)}) \\
h_{W,b} &= a_1^{(3)} = f(W_{11}^{(2)}a_1^{(2)} + W_{12}^{(2)}a_2^{(2)} + W_{13}^{(2)}a_3^{(2)} + b_1^{(2)}).
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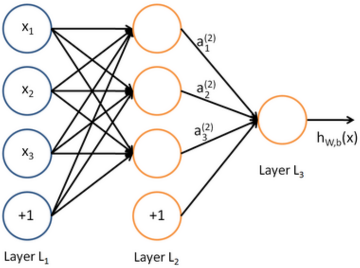
Recall (cont.)



Vector form:

$$z^{(2)} = W^{(1)}x + b^{(1)}$$
$$a^{(2)} = f(z^{(2)})$$
$$z^{(3)} = W^{(2)}a^{(2)} + b^{(2)}$$
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(average squared error with Ridge penalty).

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- As a result, we usually initialize the parameters to a small constant at random (say, using $N(0, \epsilon^2)$ for $\epsilon = 0.01$).

Gradient descent and the backpropagation algorithm

- We update the parameters using a gradient descent as follows:

$$W_{ij}^{(l)} \leftarrow W_{ij}^{(l)} - \alpha \frac{\partial}{\partial W_{ij}^{(l)}} J(W, b)$$

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$$\frac{\partial}{\partial W_{ij}^{(l)}} J(W,b) = \frac{1}{m} \sum_{i=1}^{m} \frac{\partial}{\partial W_{ij}^{(l)}} J(W,b;x^{(i)},y^{(i)}) + \lambda W_{ij}^{(l)}$$

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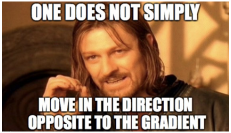
- Therefore, it suffices to compute the derivatives of $J(W,b;x^{(i)},y^{(i)})$.
- The derivatives can be recursively computed using the chain rule (the backpropagation algorithm, or backprop). See Goodfellow et al. Section 6.5.

Stochastic gradient descent and minibatches



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$$J(\theta) = \frac{1}{n} \sum_{i=1}^{n} L(x^{(i)}, y^{(i)}, \theta)$$

for some loss function L and some vector of parameters θ ($n =$ sample size).

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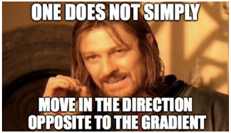

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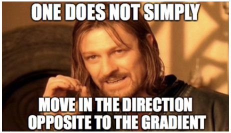

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- Thinking of $\nabla_\theta J(\theta)$ as an expected value:

$$\nabla_\theta J(\theta) \approx \frac{1}{m} \sum_{i=1}^{m} \nabla_\theta L(x^{(t_i)}, y^{(t_i)}, \theta)$$

for a subset of samples $(x^{(t_1)}, y^{(t_1)}), \ldots, (x^{(t_m)}, y^{(t_m)})$ and $1 \leq m < n$.

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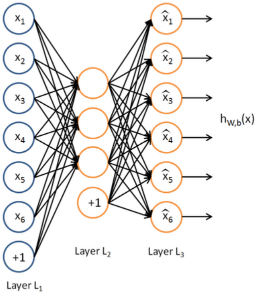
- A complete pass through the dataset is called an **epoch**.
- The optimization process is often stopped after a given number of epochs.

Autoencoders

An **autoencoder** learns the identity function:

- Input: unlabeled data.
- Output = input.
- Idea: limit the number of hidden layers to discover structure in the data.
- Learn a *compressed* representation of the input.



Source: UFLDL tutorial.

Can also learn a *sparse* network by including supplementary constraints on the weights.

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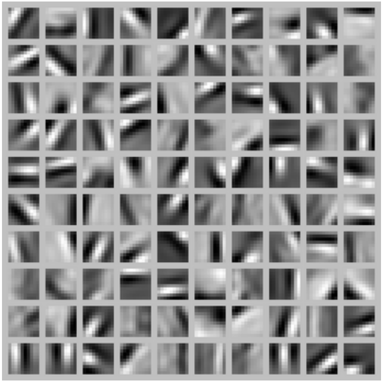
(Hint: Use Cauchy–Schwarz).

We can now display the image maximizing $a_i^{(2)}$ for each i.

Example (cont.)

100 hidden units on 10x10 pixel inputs:



The different hidden units have learned to detect edges at different positions and orientations in the image.

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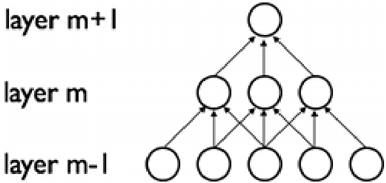

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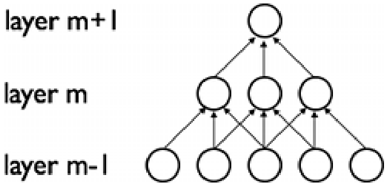
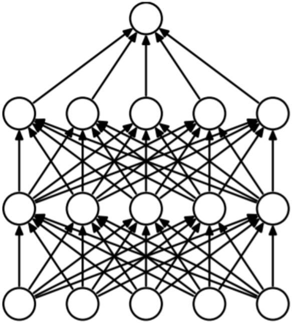
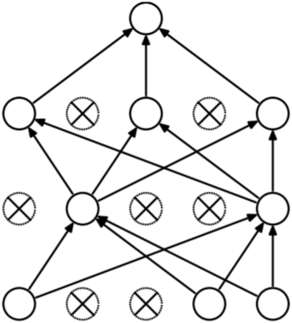


- **Dropouts:** During training, randomly ignore or ("drop out") some neurons.
- Can specify a dropout *rate* (i.e., a fixed probability $0 \leq p \leq 1$ of ignoring a given node).
- Used to learn sparse models and prevent overfitting.

Dropouts



(a) Standard Neural Net



(b) After applying dropout.

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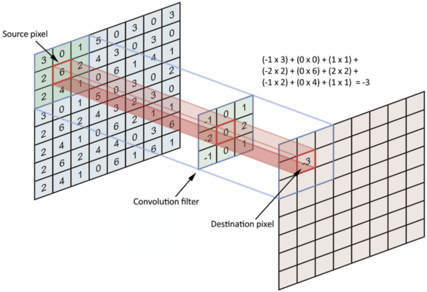
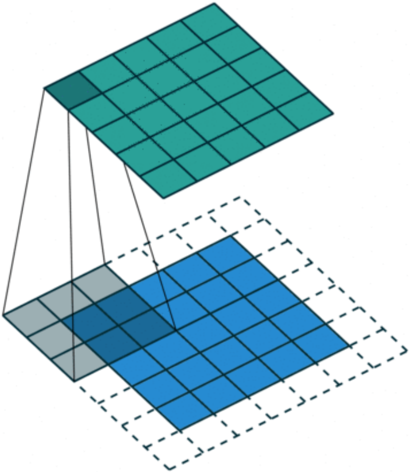


$$C(x,y) = \sum_m \sum_n I(x+m, y+m) K(m,n).$$

Example



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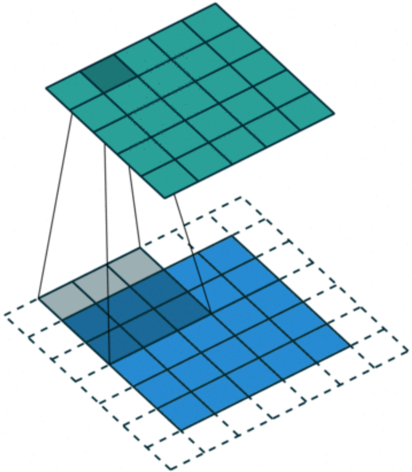

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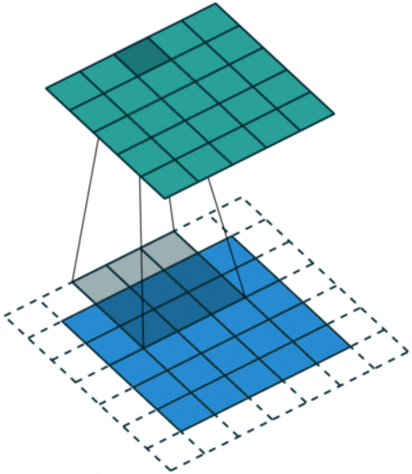

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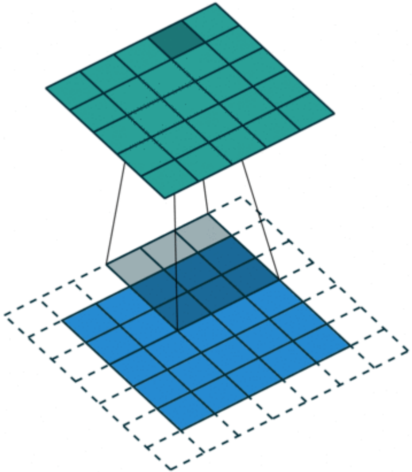

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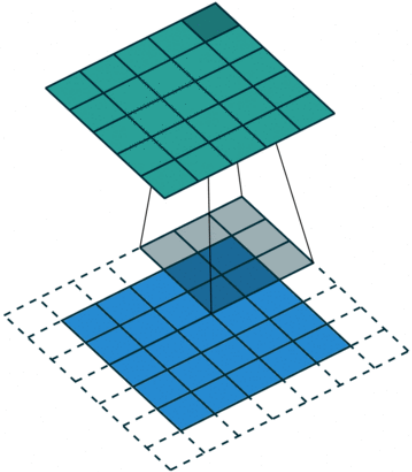
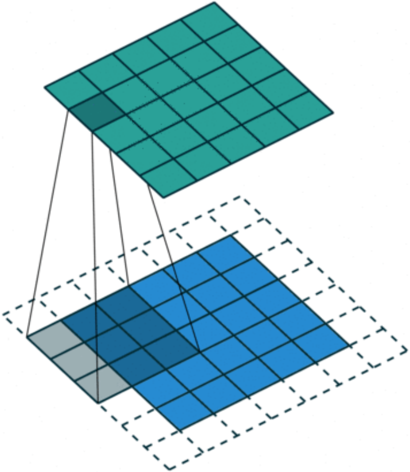
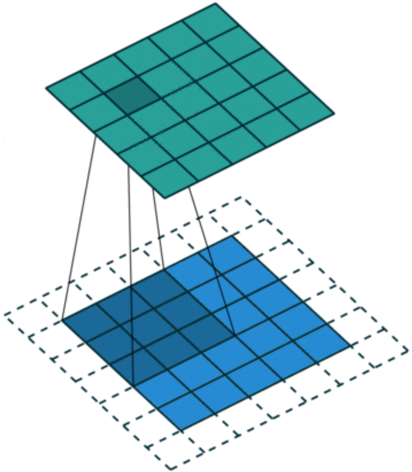




Example



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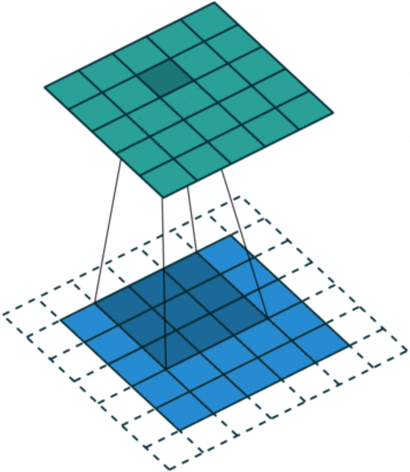

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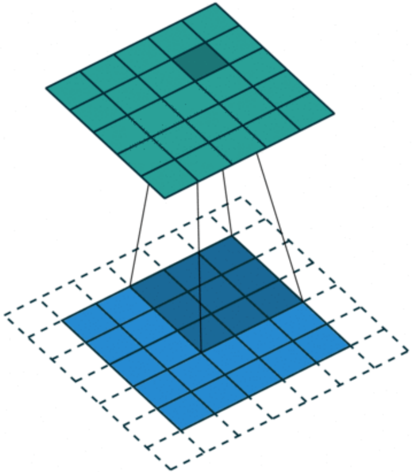

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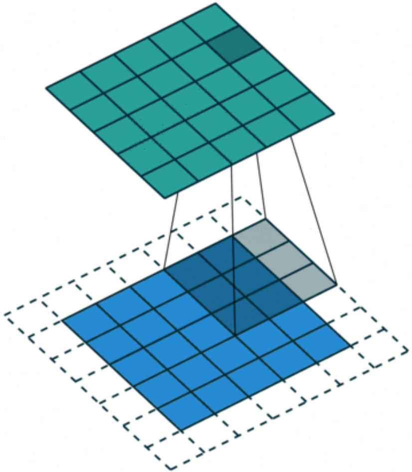

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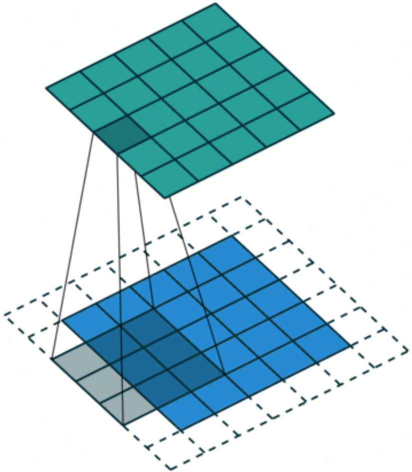

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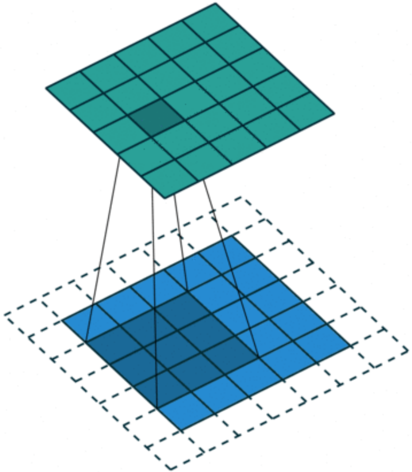

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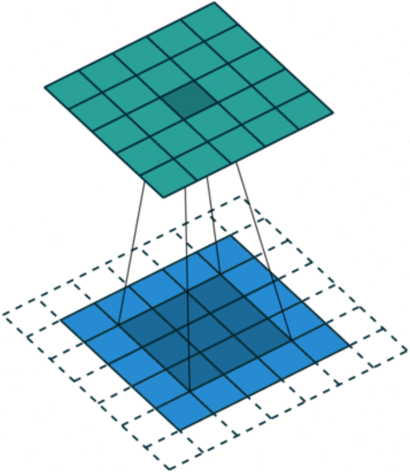

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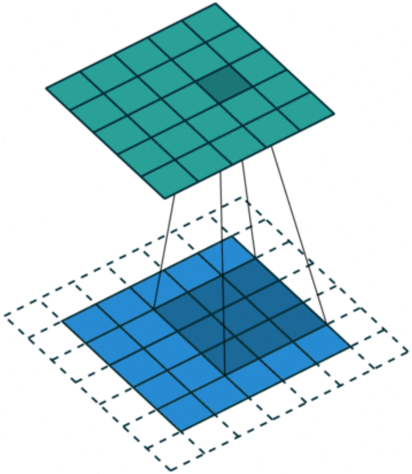

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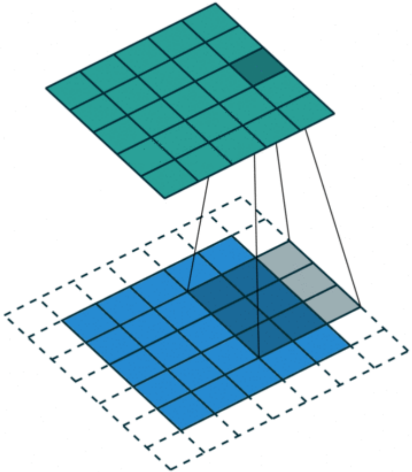

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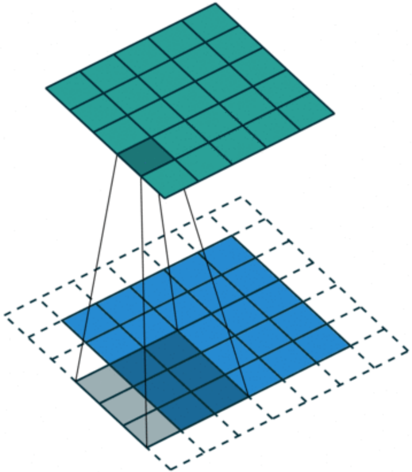

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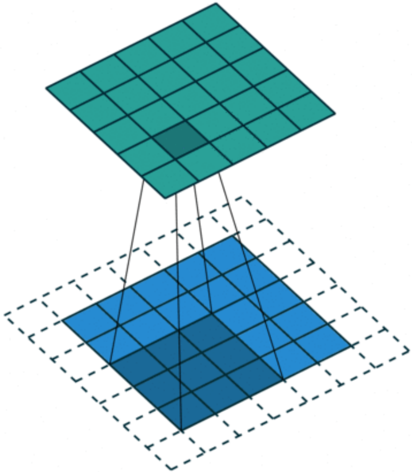

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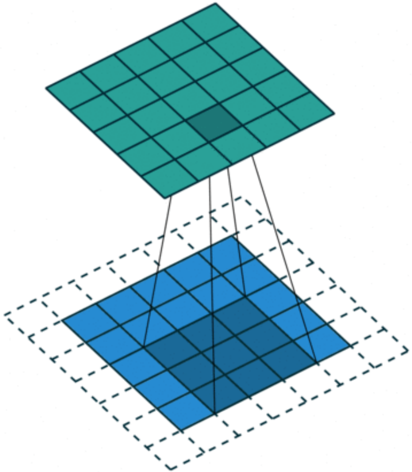

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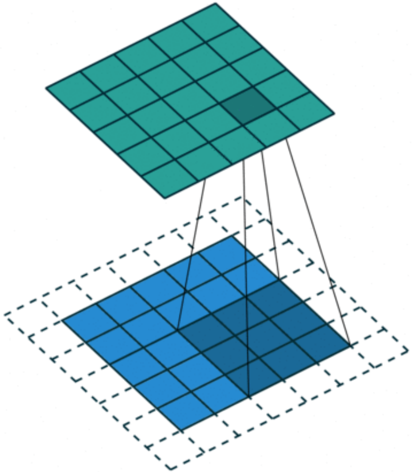

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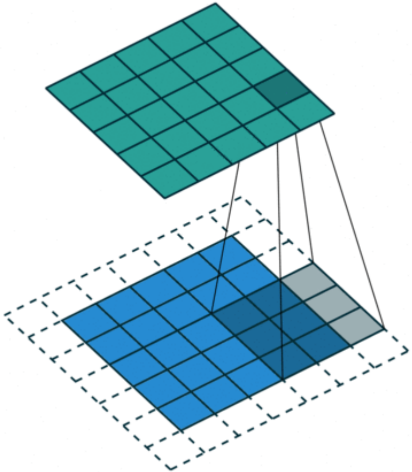

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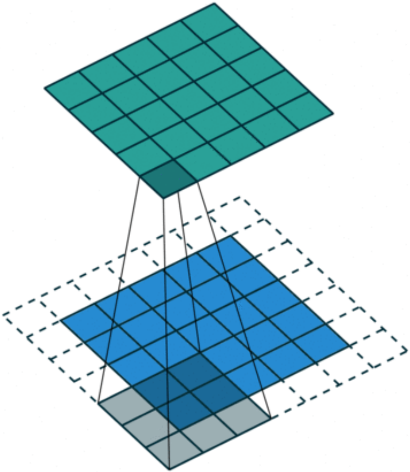

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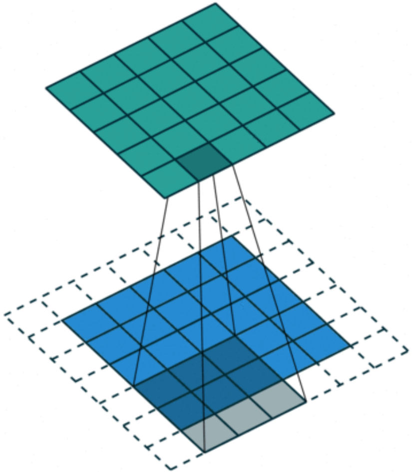

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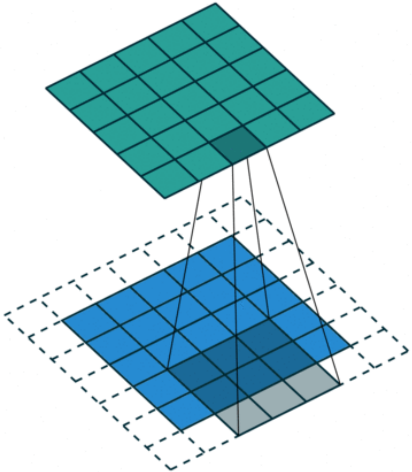

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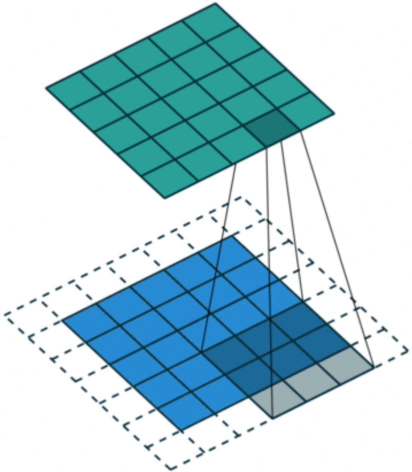
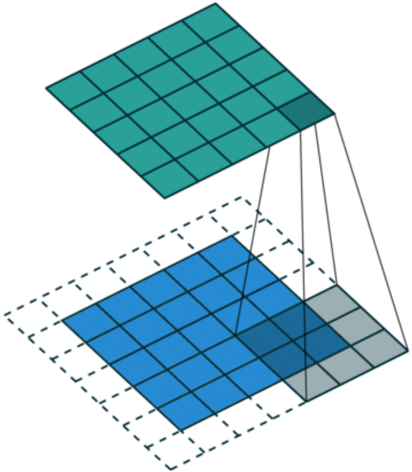


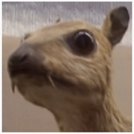
Example



Filters

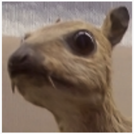
Kernel

Edge detection

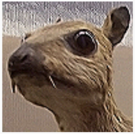


$$* \begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix} =$$

Sharpen

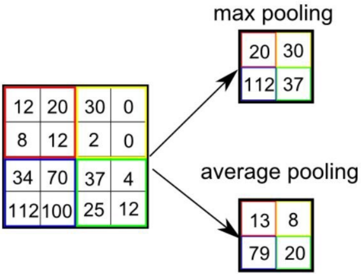


$$* \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix} =$$

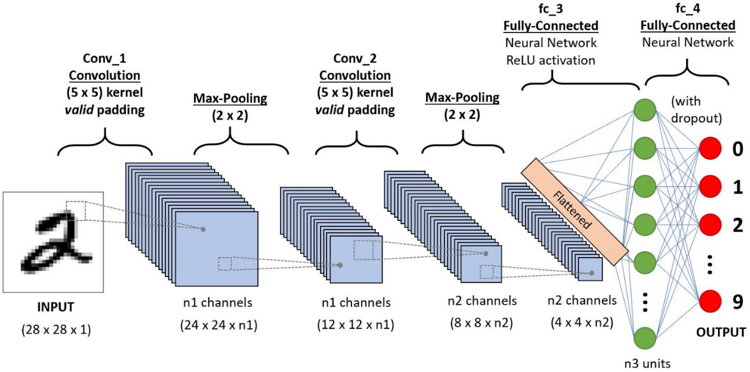


Pooling features

- Once can also *pool* the features obtained via convolution (max, mean, etc.).
- Can lead to more robust features. Can lead to invariant features.
- For example, if the pooling regions are contiguous, then the pooling units will be "translation invariant", i.e., they won't change much if objects in the image are undergo a (small) translation.



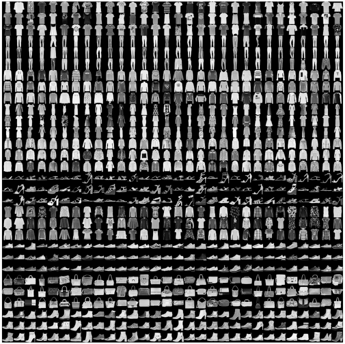
Example: handwritten digits



Homework.

Please go through (and run on your own) the tutorial available at:

`https://www.tensorflow.org/tutorials/keras/classification`



- If using Anaconda: `conda install tensorflow`.
- Can also use Google Colab:

`https://colab.research.google.com/`