MATH 637: Mathematical Techniques in Data Science
Independent component analysis

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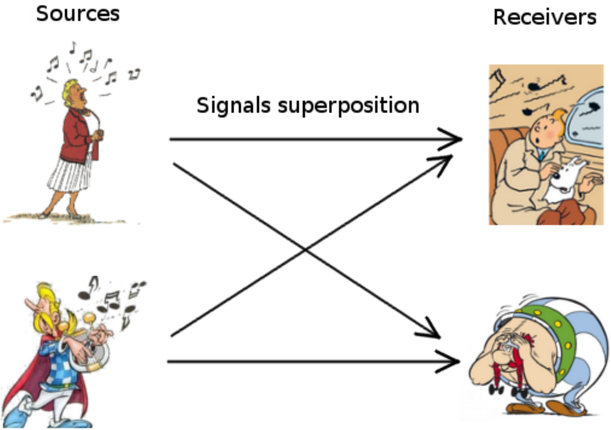
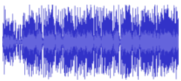
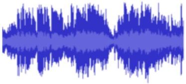


Mathematical formulation

$s_1(t)$  $s_2(t)$ 

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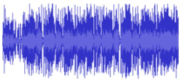
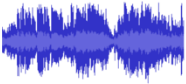
$s_1(t)$  $s_2(t)$ 

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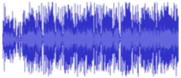
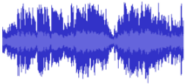
$s_1(t)$  $s_2(t)$ 

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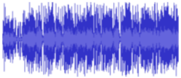
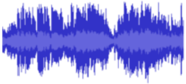
$s_1(t)$  $s_2(t)$ 

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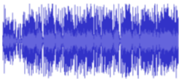
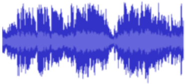
$s_1(t)$  $s_2(t)$ 

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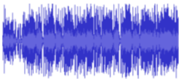
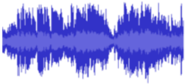
$s_1(t)$  $s_2(t)$ 

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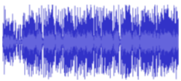
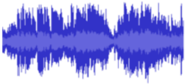
$s_1(t)$  $s_2(t)$ 

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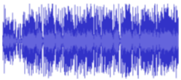
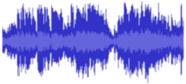
$s_1(t)$  $s_2(t)$ 

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- We will seek a solution where the components of s are *as independent as possible*.

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We will therefore assume the sources are **not** Gaussian.

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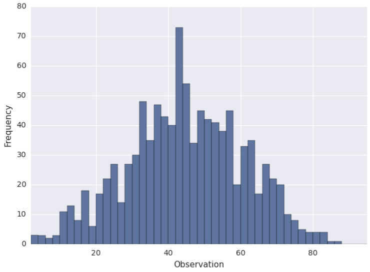

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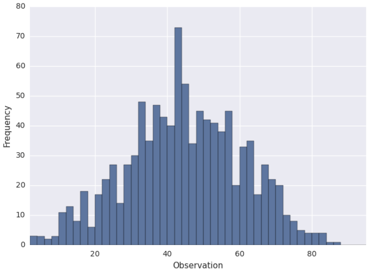

To explain the above notions, we briefly discuss some concepts from *information theory*.

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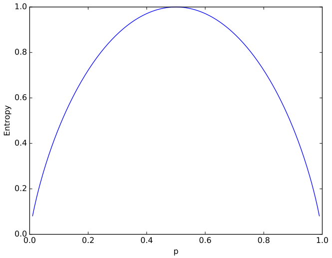
Example: If X is a (discrete) uniform on $\{1, \ldots, N\}$, then

$$H(X) = -\sum_{i=1}^{N} \frac{1}{N} \log\left(\frac{1}{N}\right) = \log N.$$

Entropy (cont.)

Example: $X \sim \text{Bernoulli}(p)$, i.e., $P(X = 1) = p$, $P(X = 0) = 1 - p$. The more "uncertain" the outcome is, the larger the entropy.



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The entropy of X is the **average information** "contained" in X:

$$H(X) = \sum_{i=1}^{N} I(p_i) p_i.$$

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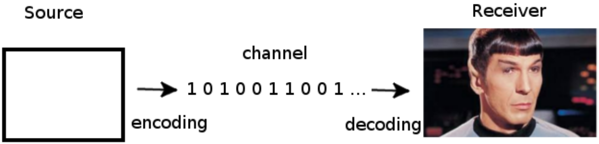

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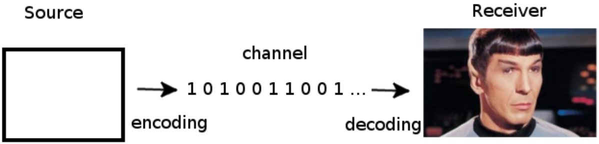

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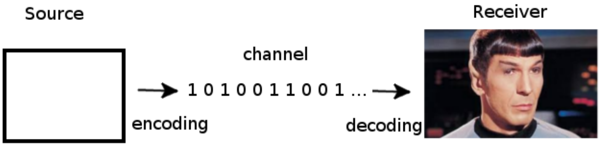

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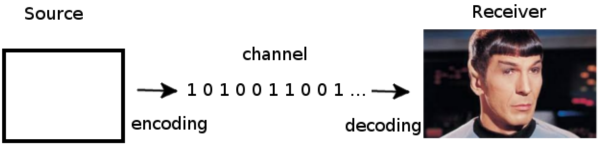

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- If the symbols an not equally likely, it is not hard to see that one can do better (i.e., send less bits per symbol on average).
- The entropy provides a **lower bound** on the average number of bits required per symbol.

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The KL divergence is used as a measure of distance between distributions (note however that $D_{\mathrm{KL}}(P||Q) \neq D_{\mathrm{KL}}(Q||P)$ in general).

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- Therefore, $I(X_1, \dots, X_n)$ provides a numerical measure of how independent random variables are.

Measures of non-Gaussianity

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  - Therefore, a variable that is "far from a Gaussian" should have a larger negentropy.

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- The negentropy is replaced by the approximation

$$J(X) \approx [E(G(X)) - E(G(X_{\text{gauss}}))]^2,$$

where $G(x) = \log\cosh(x)$.

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- Define the *whitened data matrix* by

$$X_{\text{white}} := U D^{-1/2} U^T X.$$

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The FastICA algorithm:

- Find a first direction w_1 maximizing the (approximation of) the negentropy (can use a fixed point method).
- Estimate a second direction $w_2 \perp w_1$ maximizing the (approximation of) the negentropy.
- etc..

Python - example

We mix two sound files, and recover them using ICA.

```
import scipy.io.wavfile
import numpy as np

rate, data1 = scipy.io.wavfile.read('daft-punk.wav')
rate2, data2 = scipy.io.wavfile.read('weather.wav')

mix1 = np.int16(0.3*data1+0.5*data2)[:,0]
mix2 = np.int16(0.2*data1+0.4*data2)[:,0]

scipy.io.wavfile.write('./out/mix1.wav',rate,mix1)
scipy.io.wavfile.write('./out/mix2.wav',rate,mix2)

from sklearn.decomposition import FastICA

ica = FastICA(n_components = 2)

X = np.vstack([mix1,mix2]).T

S_ = ica.fit_transform(X)
A_ = ica.mixing_

# Rescale components to have approximately the same mean amplitude as the first mixed signal
m = abs(mix1).mean()

m1 = abs(S_[:,0]).mean()
m2 = abs(S_[:,1]).mean()

S1 = np.int16(S_[:,0]*m/m1)
S2 = np.int16(S_[:,1]*m/m2)

scipy.io.wavfile.write('./out/estimated_source1.wav',rate,S1)
scipy.io.wavfile.write('./out/estimated_source2.wav',rate,S2)
```