# MATH 637: Mathematical Techniques in Data Science
# Categorical data

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March 9, 2020

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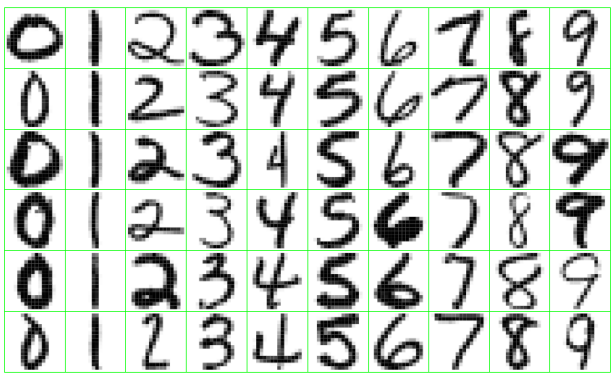
  3. Handwritten digits recognition ($Y \in \{0, \ldots, 9\}$).



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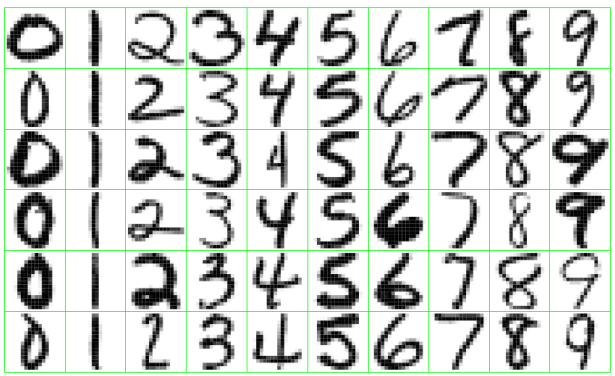

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- We begin with two very simple approaches: linear regression and nearest neighbors.

# Linear regression

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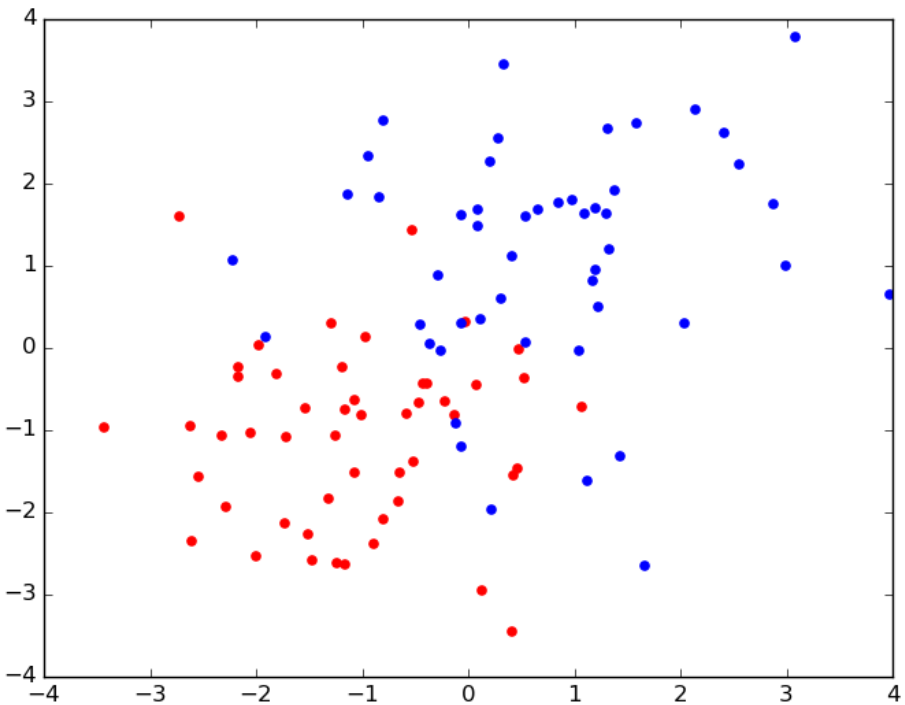
| $x$ | $y$ |
|---|---|
| (0.1,0.4) | 1 |
| (0.5,0.8) | 0 |
| (0.6, 0.2) | 1 |
| $\vdots$ | $\vdots$ |



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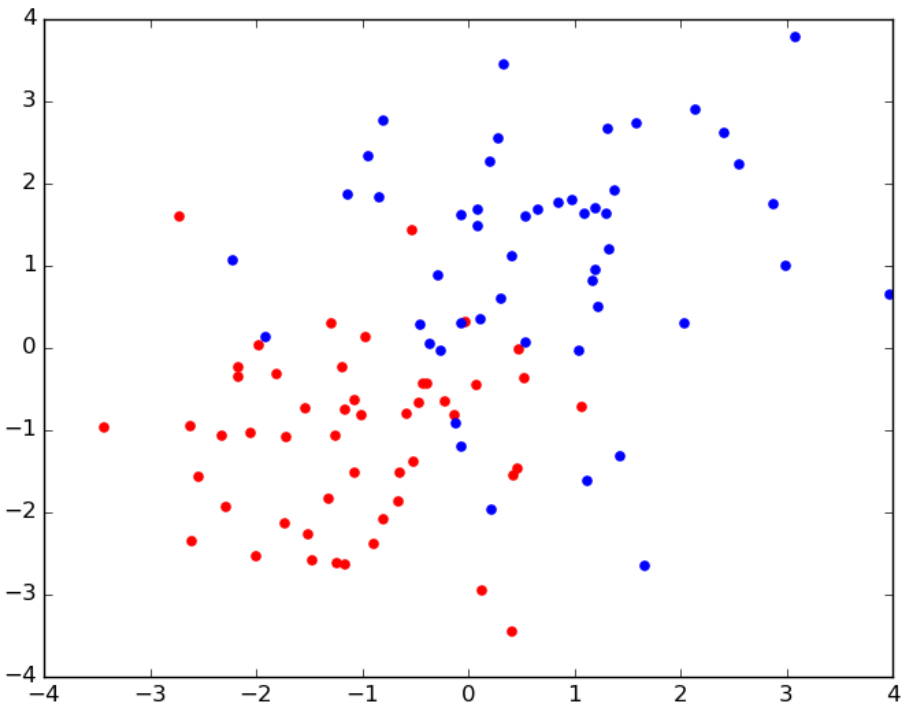


We want to predict new points' category.

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- Output is in $\{0, 1\}$, but $x^T\beta \in \mathbb{R}$.
- Use

$$\hat{y} = \begin{cases} 0 & \text{if } x^T\beta < 0.5 \\ 1 & \text{if } x^T\beta \geq 0.5 \end{cases}.$$

# Linear regression (cont.)

```
# X = 2*n by 2, Y = 2*n by 1 {0,1} labels
# Include an intercept
Xp = np.ones((2*n,3))
Xp[:,1:3] = X

# Use regression
beta = np.linalg.lstsq(Xp,Y)[0]

# Our decision boundary is
# beta[0] + beta[1] *x + beta[2]*y = 0.5,
# or y = (0.5-beta[0]-beta[1]*x)/beta[2]
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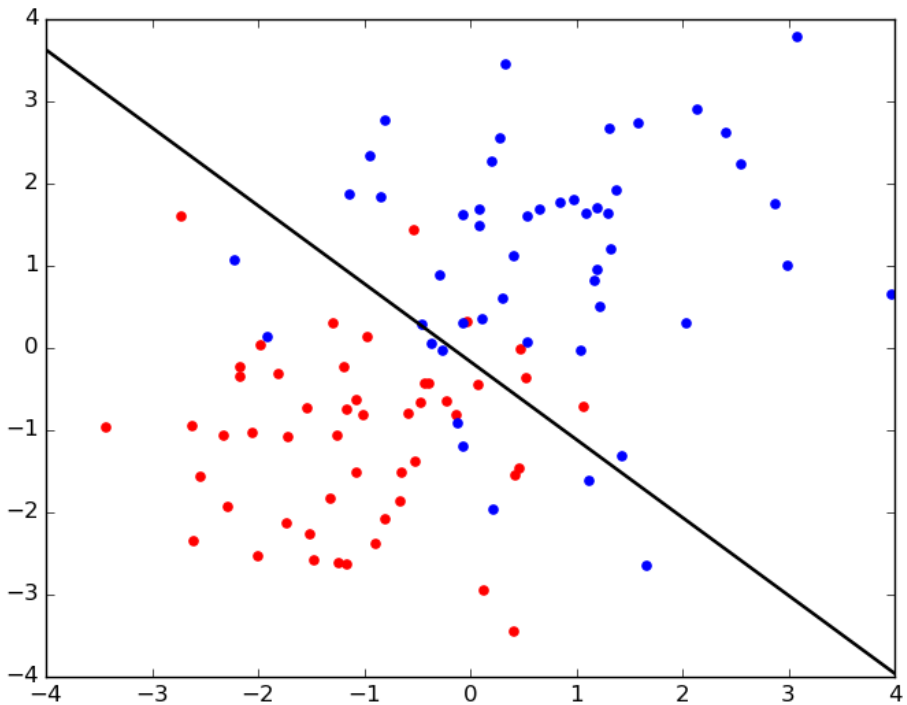
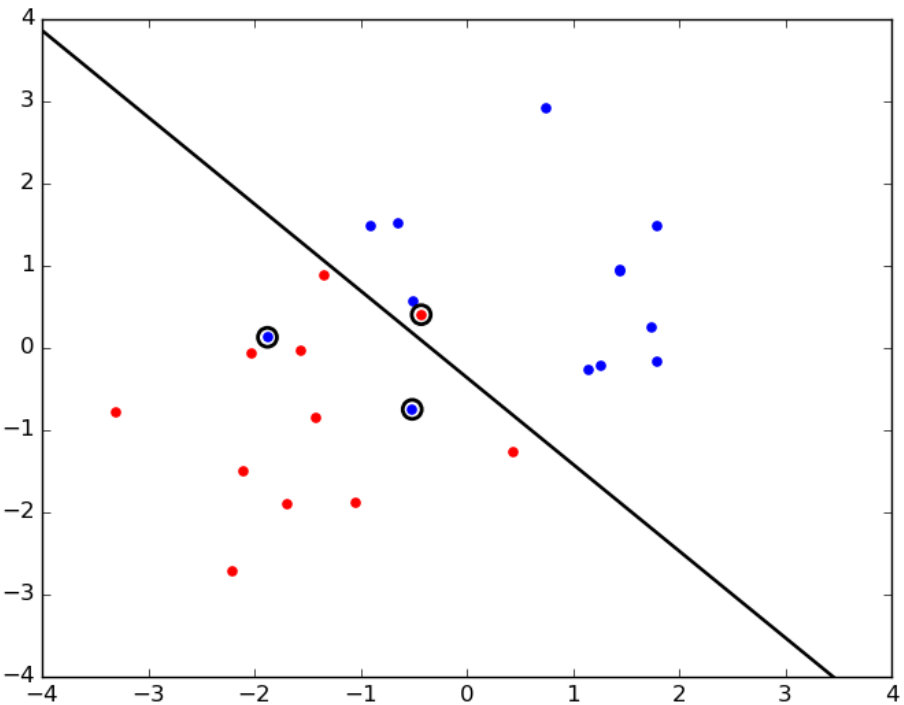


# Test error

- As usual, we split our data into train and test sets.
- Train model on training set.
- Compute classification error on test set.

```
Xp_train, Xp_test, y_train, y_test =
  train_test_split(Xp, Y, test_size=0.25)
beta = np.linalg.lstsq(Xp_train,y_train)[0]
Y_hat = Xp_test.dot(beta)
```




Test set predicted labels

In general, when working with $k$ categories, can use a loss-function

$$(L(i,j))_{i,j=1}^{k},$$

where $L(i,j) =$ cost for classifying $i$ as $j$.

# Nearest neighbors

**Nearest neighbors:** use closest observations in the training set to make predictions.

$$\hat{Y}(x) = \frac{1}{k} \sum_{x_i \in N_k(x)} y_i.$$

Here $N_k(x)$ denotes the $k$-nearest neighbors of $x$ (w.r.t. some metric, e.g. Euclidean distance).

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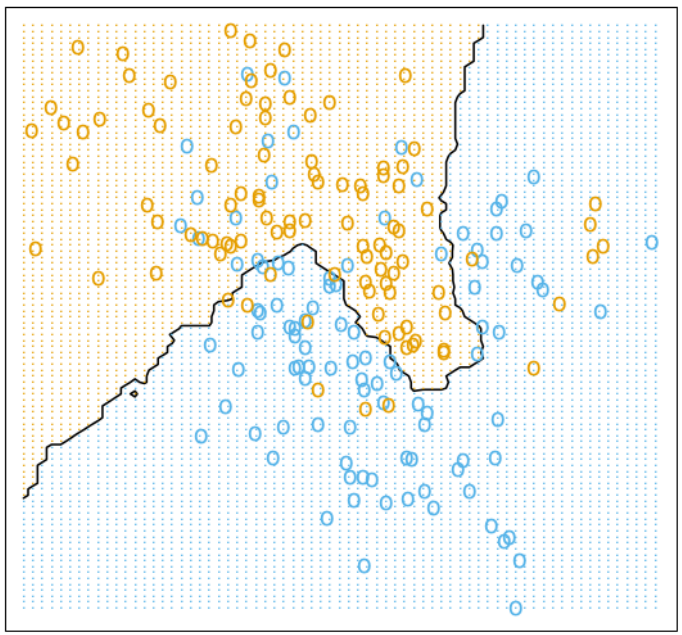
Here $N_k(x)$ denotes the $k$-nearest neighbors of $x$ (w.r.t. some metric, e.g. Euclidean distance). Use a "majority vote" to determine final labels

$$\hat{G}(x) = \begin{cases} 0 & \text{if } \hat{Y}(x) < 0.5 \\ 1 & \text{if } \hat{Y}(x) \geq 0.5 \end{cases}.$$

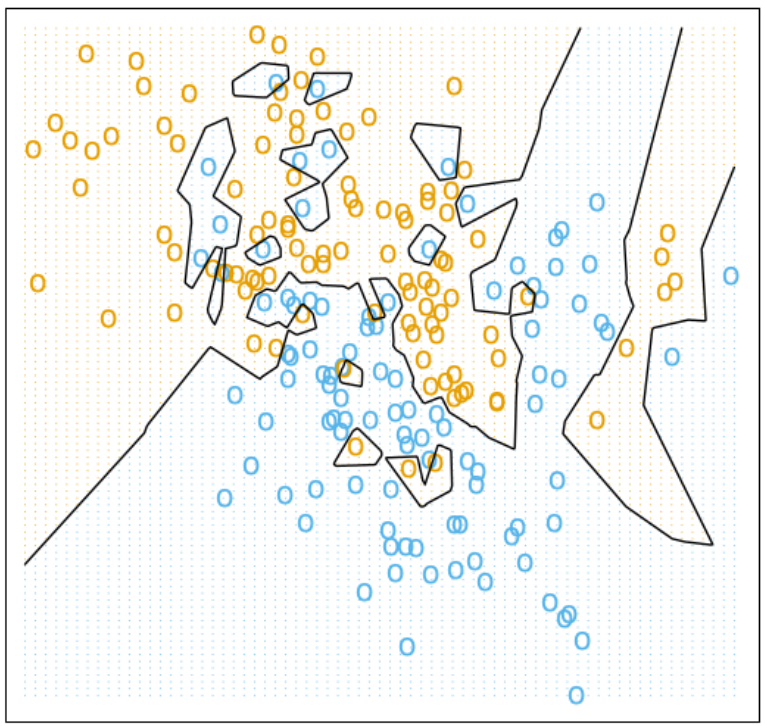


ESL, Fig. 2.2: 15 Nearest Neighbor classifier

# Nearest neighbors

Reducing the number of neighbors leads to:

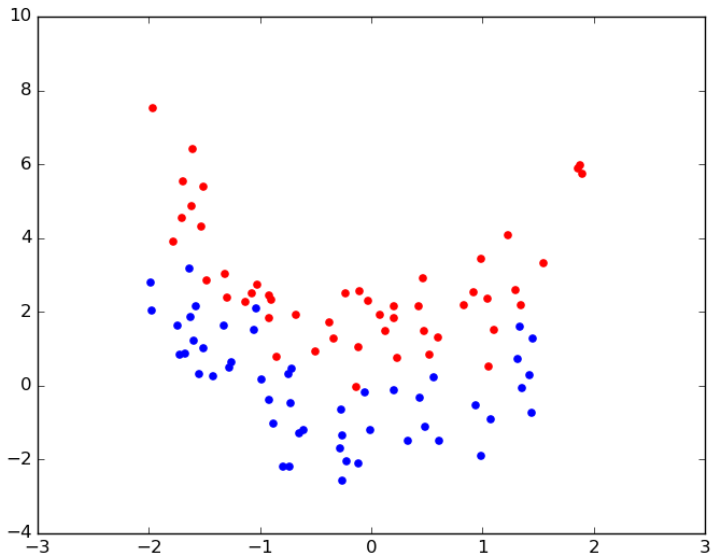
- A smaller training error (training error is $0$ when using $k = 1$ neighbor).
- Can use train/test sets to choose $k$.
- Although a small $k$ leads to a small training error, the model may not generalize well (large test error).

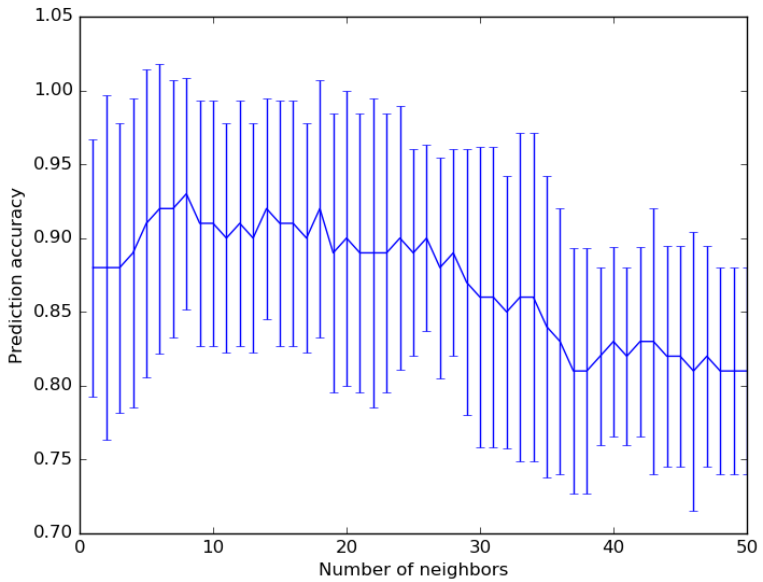


ESL, Fig. 2.3, 1 Nearest classifier

# Example

```
from sklearn.neighbors import KNeighborsClassifier
neigh = KNeighborsClassifier(n_neighbors=i)
```




```
for i in range(1,N_max+1):
    neigh = KNeighborsClassifier(n_neighbors=i)
    for j, (train, test) in enumerate(kf):
        X_train, X_test, y_train, y_test =  X[train], X[test],
                                            Y[train], Y[test]
        neigh.fit(X_train,y_train)
        scores[i-1,j] = neigh.score(X_test, y_test)
```

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Each method has its own situations for which it works best
Both methods can lead to very good predictions.
Many strategies exist to improve these methods (as we will see later).