MATH 567: Mathematical Techniques in Data Science
Lab 2

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Exercise

1. Load the `Auto` dataset.
2. Use the `lm` function to fit a linear model $\texttt{mpg} = \beta_0 + \beta_1 \cdot \texttt{horsepower} + \beta_2 \cdot \texttt{weight}$.
3. Compute the coefficients directly by solving the normal equations. Do you get the same results?

Note: You may need to convert the data frame to a matrix using `as.matrix(X)`.

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If you do not get the same results: did you include an intercept in the normal equations?

```
X = as.matrix(Auto[,c(4,5)])
Xp = cbind(matrix(1,392,1), X)
```

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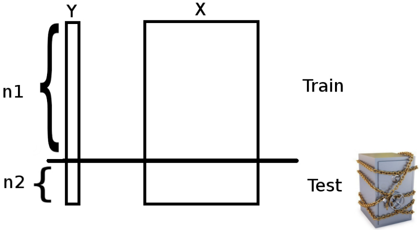
3. We use the fitted model to predict values of the test data and compute the **test error**.

Splitting data into training/test data:



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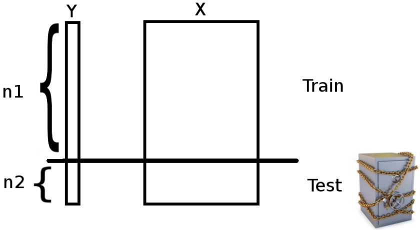

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1. $\hat{\beta} = (X_{\text{train}}^T X_{\text{train}})^{-1} X_{\text{train}}^T Y_{\text{train}}$.

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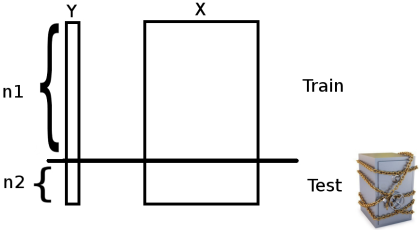

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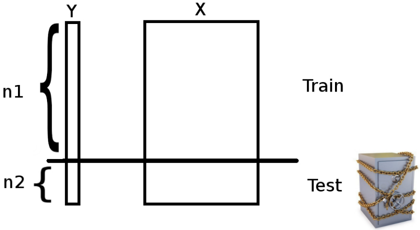

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$$\text{MSE}_{\text{test}} = \frac{1}{n_2} \sum_{i=1}^{n_2} (\widehat{Y}_{\text{test},i} - Y_{\text{test},i})^2.$$

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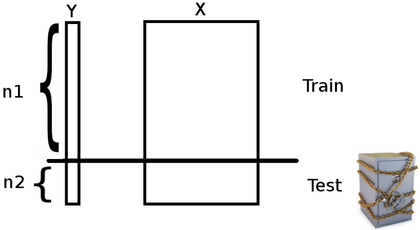

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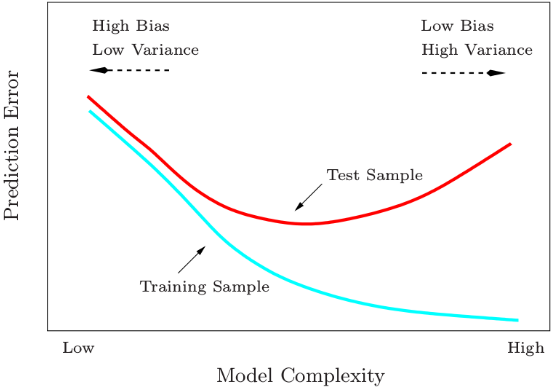
We choose a model that minimizes the test error.

Training error and test error (cont.)

Typical behavior of the test and training error, as model complexity is varied.



ESL, Fig 2.11.

Train/test sets in R

```
library(ISLR)
data(Auto)

Auto <- Auto[,-9]  # Remove the "names" column

n <- dim(Auto)[1]

ntrain <- floor(0.75*n)
ntest <- n - ntrain

train_ind <-  sample(1:n, ntrain)

train <-  Auto[train_ind,]
test <-  Auto[-train_ind,]
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Compute the test error:

```
model_full <- lm(mpg ~ ., data=train)
mean((predict(model_full, test[,-1]) - test[,1])**2)
```

Using a subset of variables

Fit a model using only the last 3 variables:

```
model <- lm(mpg ~ ., data=train[,append(c(5,7,8),1)])
mean((predict(model, test[,c(5,7,8)]) - test[,1])**2)
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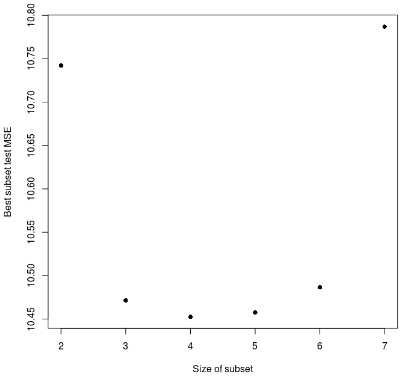
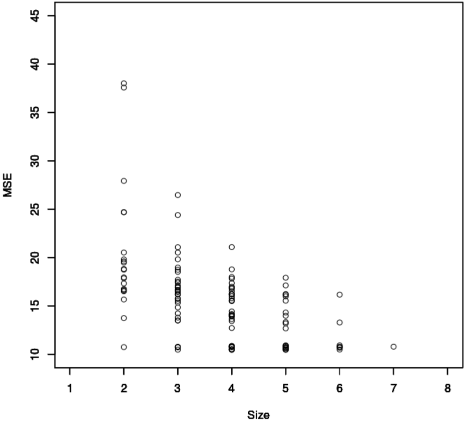
Minimal test error for subsets of a given size:



Examining all subsets

For this dataset, we can examine all the possible subsets (usually impossible):



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- The leaps and bounds procedure (Furnival and Wilson, 1974) makes this feasible for p as large as 30 or 40.

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Nevertheless, the stepwise approaches often return predictors similar to the predictors obtained from more complex methods with better theory.

The R package leaps

1. Install and load the `leaps` package.
2. Use the `regsubsets` function to perform forward and backward stepwise regressions.

```
library(leaps)

regfit.fwd = regsubsets(mpg ~ ., data=Auto[,-9],
   method="forward")

regfit.bwd = regsubsets(mpg ~ ., data=Auto[,-9],
   method="backward")
```

3. Examine the output of `summary(regfit.fwd)` and `plot(regfit.fwd)`.