MATH 567: Mathematical Techniques in Data Science
Lab 8

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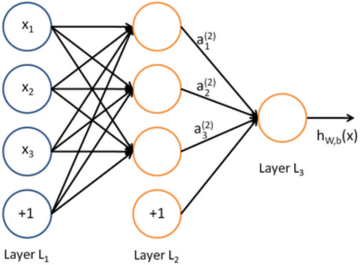
April 11, 2017

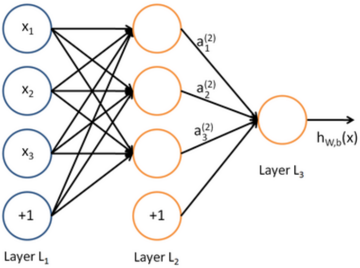
Recall



We have:

$$a_1^{(2)} = f(W_{11}^{(1)} x_1 + W_{12}^{(1)} x_2 + W_{13}^{(1)} x_3 + b_1^{(1)})$$
$$a_2^{(2)} = f(W_{21}^{(1)} x_1 + W_{22}^{(1)} x_2 + W_{23}^{(1)} x_3 + b_2^{(1)})$$
$$a_3^{(2)} = f(W_{31}^{(1)} x_1 + W_{32}^{(1)} x_2 + W_{33}^{(1)} x_3 + b_3^{(1)})$$
$$h_{W,b} = a_1^{(3)} = f(W_{11}^{(2)} a_1^{(2)} + W_{12}^{(2)} a_2^{(2)} + W_{13}^{(2)} a_3^{(2)} + b_1^{(2)}).$$

Recall (cont.)



Vector form:

$$
\begin{aligned}
z^{(2)} &= W^{(1)}x + b^{(1)} \\
a^{(2)} &= f(z^{(2)}) \\
z^{(3)} &= W^{(2)}a^{(2)} + b^{(2)} \\
h_{W,b} &= a^{(3)} = f(z^{(3)}).
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(average squared error with Ridge penalty).

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Note:

- The Ridge penalty prevents overfitting.
- We do not penalize the bias terms $b_i^{(l)}$.

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- We need an initial choice for $W_{ij}^{(l)}$ and $b_i^{(l)}$. If we initialize all the parameters to 0, then the parameters remain constant over the layers because of the symmetry of the problem.
- As a result, we initialize the parameters to a small constant at random (say, using $N(0, \epsilon^2)$ for $\epsilon = 0.01$).

- We update the parameters using a gradient descent as follows:

$$W_{ij}^{(l)} \leftarrow W_{ij}^{(l)} - \alpha \frac{\partial}{\partial W_{ij}^{(l)}} J(W, b)$$

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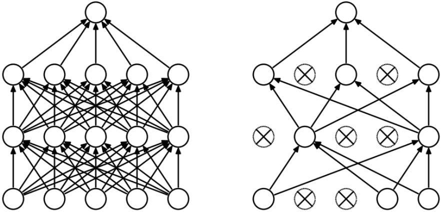
- The partial derivatives can be cleverly computed using the chain rule to avoid repeating calculations (backpropagation algorithm).

Sparse neural networks

Sparse networks can be built by

- Penalizing coefficients (e.g. using a ℓ_1 penalty).
- Dropping some of the connections at random (dropout).



Srivastava et al., JMLR 15 (2014).

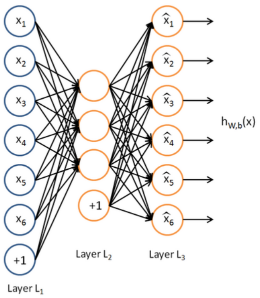
Useful to prevent overfitting.

Recent work: "One-shot learners" can be used to train models with a smaller sample size.

Autoencoders

An **autoencoder** learns the identity function:

- Input: unlabeled data.
- Output = input.
- Idea: limit the number of hidden layers to discover structure in the data.
- Learn a *compressed* representation of the input.



Source: UFLDL tutorial.

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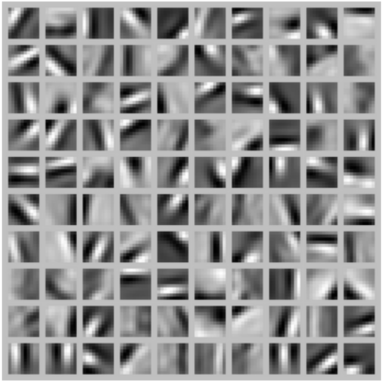
(Hint: Use Cauchy–Schwarz).

We can now display the image maximizing $a_i^{(2)}$ for each i.

Example (cont.)

100 hidden units on 10x10 pixel inputs:



The different hidden units have learned to detect edges at different positions and orientations in the image.

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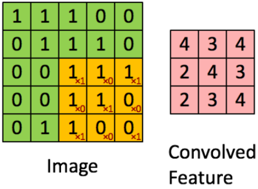

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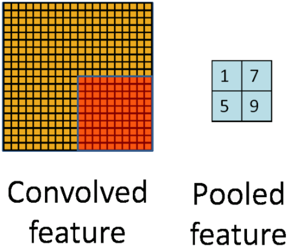

We will use the package h2o to train neural networks with R. To get you started, we will construct a neural network with 1 hidden layers containing 2 neurons to learn the XOR function:

	0	1
0	0	1
1	1	0

```
# Initialize h2o
library(h2o)

h2o.init(nthreads=-1, max_mem_size="2G")
h2o.removeAll() # in case the cluster was
                # already running

# Construct the XOR function
X = t(matrix(c(0,0,0,1,1,0,1,1),2,4))
y = matrix(c(-1,1,1,-1), 4)
train = as.h2o(cbind(X,y))
```

R (cont.)

Training the model:

```
# Train model
model <- h2o.deeplearning(x = names(train)[1:2],
                          y = names(train)[3],
                          training_frame = train,
                          activation = "Tanh",
                          hidden = c(2),
                          input_dropout_ratio = 0.0,
                          l1 = 0,
                          epochs = 10000)

# Test the model
h2o.predict(model, train)
```

Some options you may want to use when building more complicated models for data:

```
activation = "RectifierWithDropout"
input_dropout_ratio = 0.2
l1 = 1e-5
```