MATH 567: Mathematical Techniques in Data Science
Support vector machines

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Hyperplanes

Recall:

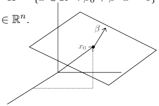
- A *hyperplane* H in $V = \mathbb{R}^n$ is a subspace of V of dimension $n-1$ (i.e., a subspace of codimension 1).
- Each hyperplane is determined by a nonzero vector $\beta \in \mathbb{R}^n$ via

$$H = \{x \in \mathbb{R}^n : \beta^T x = 0\} = \text{span}(\beta)^\perp.$$

- An *affine hyperplane* H in $\mathbb{R}^n$ is a subset of the form

$$H = \{x \in \mathbb{R}^n : \beta_0 + \beta^T x = 0\}$$

where $\beta_0 \in \mathbb{R}, \beta \in \mathbb{R}^n$.

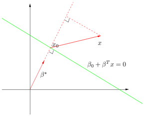


- We often use the term "hyperplane" for "affine hyperplane".

Hyperplanes (cont.)

Let

$$H = \{x \in \mathbb{R}^n : \beta_0 + \beta^T x = 0\}.$$



Note that for $x_0, x_1 \in H$,

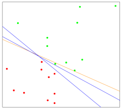
$$\beta^T (x_0 - x_1) = 0.$$

Thus β is perpendicular to H. It follows that for $x \in \mathbb{R}^n$,

$$d(x, H) = \frac{\beta^T}{\|\beta\|}(x - x_0) = \frac{\beta_0 + \beta^T x}{\|\beta\|}.$$

Separating hyperplane

Suppose we have binary data with labels $\{+1, -1\}$. We want to separate data using an (affine) hyperplane.



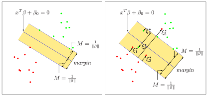
ESL, Figure 4.14. (Orange = least-squares)

Classify using $G(x) = \text{sgn}(x^T \beta + \beta_0)$.

- Separating hyperplane may not be unique.
- Separating hyperplane may not exist (i.e., data may not be separable).

Margins

Uniqueness problem: when the data is separable, choose the hyperplane to maximize the "margin" (the "no man's land").



Data: $(y_i, x_i) \in \{+1, -1\} \times \mathbb{R}^p \qquad (i = 1, \dots, n)$.
Suppose $\beta_0 + \beta^T x$ is a separating hyperplane with $\|\beta\| = 1$.
Note that:

$$y_i(x_i^T\beta + \beta_0) > 0 \Rightarrow \text{Correct classification}$$
$$y_i(x_i^T\beta + \beta_0) < 0 \Rightarrow \text{Incorrect classification}$$

Also, $|y_i(x_i^T\beta + \beta_0)| =$ distance between x and hyperplane (since $\|\beta\| = 1$).

Margins (cont.)

Thus, if the data is separable, we can solve

$$\max_{\beta_0, \beta \in \mathbb{R}^p, \|\beta\|=1} M$$
$$\text{subject to } y_i(x_i^T\beta + \beta_0) \geq M \qquad (i = 1, \dots, n).$$

We will transform the problem into a usual form used in convex optimization.
We can remove $\|\beta\| = 1$ by replacing the constraint by

$$\frac{1}{\|\beta\|} y_i(x_i^T\beta + \beta_0) \geq M, \quad \text{or equivalently,} \quad y_i(x_i^T\beta + \beta_0) \geq M\|\beta\|.$$

We can always rescale (β, β_0) so that $\|\beta\| = 1/M$. Our problem is therefore equivalent to

$$\min_{\beta_0, \beta \in \mathbb{R}^p} \frac{1}{2}\|\beta\|^2$$
$$\text{subject to } y_i(x_i^T\beta + \beta_0) \geq 1 \qquad (i = 1, \dots, n).$$

We now recognize the problem as a convex optimization problem with a quadratic objective, and linear inequality constraints.

Support vector machines

- The previous problem works well when the data is *separable*. What happens if there is no way to find a margin?
- We allow some points to be on the wrong side of the margin, but keep control on the error. We replace $y_i(x_i^T\beta + \beta_0) \geq M$ by

$$y_i(x_i^T\beta + \beta_0) \geq M(1 - \xi_i), \qquad \xi_i \geq 0,$$

and add the constraint

$$\sum_{i=1}^n \xi_i \leq C \qquad \text{for some fixed constant } C > 0.$$

The problem becomes:

$$\max_{\beta_0, \beta \in \mathbb{R}^p, \|\beta\|=1} M$$
$$\text{subject to } y_i(x_i^T\beta + \beta_0) \geq M(1 - \xi_i)$$
$$\xi_i \geq 0, \qquad \sum_{i=1}^n \xi_i \leq C.$$

Support vector machines (cont.)

As before, we can transform the problem into its "normal" form:

$$\min_{\beta_0, \beta} \frac{1}{2}\|\beta\|^2$$
$$\text{subject to } y_i(x_i^T\beta + \beta_0) \geq 1 - \xi_i$$
$$\xi_i \geq 0, \qquad \sum_{i=1}^n \xi_i \leq C.$$

Problem can be solved using standard optimization packages.