MATH 829: Introduction to Data Mining and Analysis
Neural networks II

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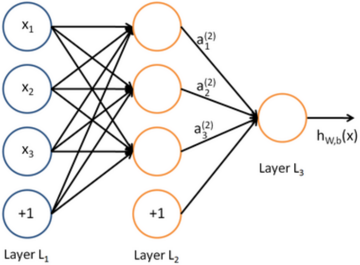
This lecture is based on the UFLDL tutorial (http://deeplearning.stanford.edu/)

Recall



We have:

$$
\begin{aligned}
a_1^{(2)} &= f(W_{11}^{(1)} x_1 + W_{12}^{(1)} x_2 + W_{13}^{(1)} x_3 + b_1^{(1)}) \\
a_2^{(2)} &= f(W_{21}^{(1)} x_1 + W_{22}^{(1)} x_2 + W_{23}^{(1)} x_3 + b_2^{(1)}) \\
a_3^{(2)} &= f(W_{31}^{(1)} x_1 + W_{32}^{(1)} x_2 + W_{33}^{(1)} x_3 + b_3^{(1)}) \\
h_{W,b} &= a_1^{(3)} = f(W_{11}^{(2)} a_1^{(2)} + W_{12}^{(2)} a_2^{(2)} + W_{13}^{(2)} a_3^{(2)} + b_1^{(2)}).
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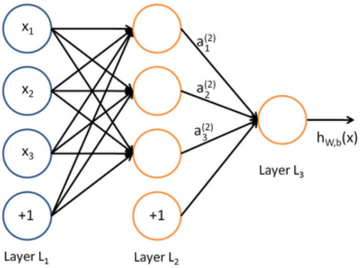

Vector form:

$$z^{(2)} = W^{(1)}x + b^{(1)}$$
$$a^{(2)} = f(z^{(2)})$$
$$z^{(3)} = W^{(2)}a^{(2)} + b^{(2)}$$
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- As a result, we initialize the parameters to a small constant at random (say, using $N(0, \epsilon^2)$ for $\epsilon = 0.01$).

Gradient descent and the backpropagation algorithm

We update the parameters using a gradient descent as follows:

$$W_{ij}^{(l)} \leftarrow W_{ij}^{(l)} - \alpha \frac{\partial}{\partial W_{ij}^{(l)}} J(W, b)$$
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Observe that:

$$\frac{\partial}{\partial W_{ij}^{(l)}} J(W,b) = \frac{1}{m} \sum_{i=1}^{m} \frac{\partial}{\partial W_{ij}^{(l)}} J(W,b;x^{(i)},y^{(i)}) + \lambda W_{ij}^{(l)}$$

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Therefore, it suffices to compute the derivatives of $J(W,b;x^{(i)},y^{(i)})$.

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3. For $l = n_l - 1, n_l - 2, \ldots, 2$
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4. Compute the desired partial derivatives:

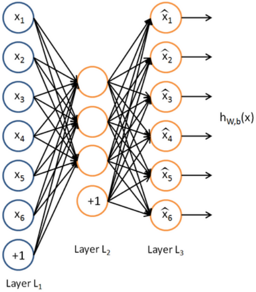
$$\frac{\partial}{\partial W_{ij}^{(l)}} J(W, b; x^{(i)}, y^{(i)}) = a_j^{(l)} \delta_i^{(l+1)}$$

$$\frac{\partial}{\partial b_i^{(l)}} J(W, b; x^{(i)}, y^{(i)}) = \delta_i^{(l+1)}.$$

Autoencoders

An **autoencoder** learns the identity function:

- Input: unlabeled data.
- Output = input.
- Idea: limit the number of hidden layers to discover structure in the data.
- Learn a *compressed* representation of the input.



Source: UFLDL tutorial.

Can also learn a *sparse* network by including supplementary constraints on the weights.

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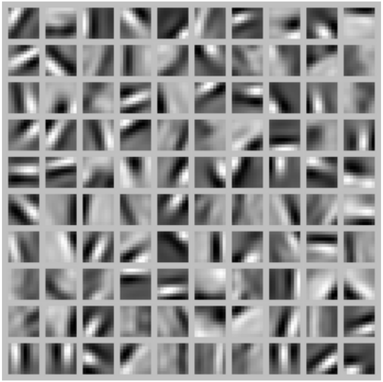
(Hint: Use Cauchy–Schwarz).

We can now display the image maximizing $a_i^{(2)}$ for each i.

Example (cont.)

100 hidden units on 10x10 pixel inputs:



The different hidden units have learned to detect edges at different positions and orientations in the image.

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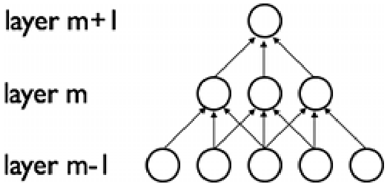
- Dense networks have a lot of parameters to learn. Can be inefficient or impossible to train.
- *Sparse* models have been proposed in the literature.
- Some of these models find inspiration from how the early visual system is wired up in biology.



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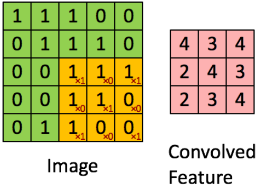

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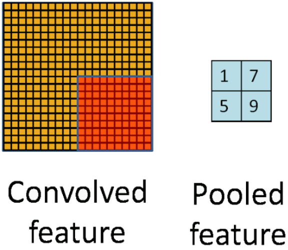

Neural networks with scikit-learn

Need to install the 0.18-dev version (`http://scikit-learn.org/stable/developers/contributing.html#retrieving-the-latest-code`).

- `sklearn.neural_network.MLPClassifier`
- `sklearn.neural_network.MLPRegressor`