MATH 829: Introduction to Data Mining and Analysis
The EM algorithm

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Missing values in data

Missing data is a common problem in statistics.

- No measurement for a given individual/time/location, etc.
- Device failed.
- Error in data entry.
- Data was not disclosed for privacy reasons.
- etc.

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Andersson, Mr. Anders Johan	male	39.0	1
Vestrom, Miss. Hulda Amanda Adolfina	female	14.0	0
Hewlett, Mrs. (Mary D Kingcome)	female	55.0	0
Rice, Master. Eugene	male	2.0	4
Williams, Mr. Charles Eugene	male	NaN	0
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Missing values in the titanic passengers dataset.

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How can we deal with missing values?

- Many possible procedures.
- The choice of the procedure can significantly impact the conclusions of a study.

Some strategies for dealing with missing values

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- **Imputation with the EM algorithm.**
  Replace missing values by the *most likely values*. Account for all information available. Much more rigorous. However, requires a model. Can be computationally intensive.

Missing data mechanism

"Types" of missing data:

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- Some respondent may not answer the survey for no particular reason. MCAR
- Maybe women are less likely to answer than male (independently of their weight). MAR
- Heavy or light people may be less likely to disclose their weight. MNAR.

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- Ignoring the missing data mechanism, we have

$$p(x_1, \mathrm{NA}, x_3, x_4) = \sum_{x=0}^{3} p(x_1, x, x_2, x_3).$$

Example (cont.)

- Suppose the data comes from a parametric model $p(x_1, x_2, x_3, x_4; \theta)$ where $\theta \in \Theta$ is unknown.

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- We compute the *likelihood* of the data:

$$L(\theta) = p(2,0,2,3) \times p_{1,3,4}(3,1,1) \times p_{1,2}(1,3) \times p_{1,3}(2,1),$$

where $p_{1,3,4}(x_1, x_3, x_4) = \sum_{x_2=0}^{3} p(x_1, x_2, x_3, x_4)$,
$p_{1,2}(x_1, x_2) = \sum_{x_3=0}^{3} \sum_{x_4=0}^{3} p(x_1, x_2, x_3, x_4)$, and
$p_{1,3}(x_1, x_3) = \sum_{x_2=0}^{3} \sum_{x_4=0}^{3} p(x_1, x_2, x_3, x_4)$ denote *marginals* of p.

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- The *likelihood* can now be maximized as a function of θ.

Imputing the missing values

- Recall that $f(x) = E(Y|X = x)$ has the following optimality property:

$$E(Y|X = x) = \underset{c \in \mathbb{R}}{\operatorname{argmin}} \, E(Y - c)^2$$

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For example, if $x = (1, 3, \text{NA}, \text{NA})$ then:
$$(\hat{x}_3, \hat{x}_4) = E((X_3, X_4)|X_1 = 1, X_2 = 3),$$
where E is computed with respect to $p(x_1, x_2, x_3, x_4; \theta)$.

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Remark: We assumed above that the variables are discrete, and the observations are independent for simplicity. The same procedure applied in the general case.

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The EM algorithm leverages the fact the the likelihood is often easy to maximize if there is no missing values.

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- We would like to maximize that function over θ (generally difficult).

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Remark: There is no guarantee that the EM algorithm will find the **global** max of the likelihood. It may only find a **local** max.