MATH 829: Introduction to Data Mining and Analysis
Graphical Models I

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May 2, 2016

Independence and conditional independence: motivation

We begin with a classical example (Whittaker, 1990):

- We study the examination marks of 88 students in five subjects: mechanics, vectors, algebra, analysis, statistics (Mardia, Kent, and Bibby, 1979).
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20-	1	012455699
30-	1266677889	0011122333344556677799
40-	0113333345566667777888999999	0000000112344455556679
50-	000000111223333344455666778899	0113346
60-	000111123455578	11233447888
70-	12	033
80-	0	1111
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Note: Data appears to be normally distributed.

Example (cont.)

We compute the correlation between the grades of the students:

mech	1.0				
vect	0.55	1.0			
alg	0.55	0.61	1.0		
anal	0.41	0.49	0.71	1.0	
stat	0.39	0.44	0.66	0.61	1.0
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We now examine the inverse correlation matrix:

mech	1.60				
vect	-0.56	1.80			
alg	-0.51	-0.66	3.04		
anal	**0.00**	**-0.15**	-1.11	2.18	
stat	**-0.04**	**-0.04**	-0.86	-0.52	1.92
	mech	vect	alg	anal	stat

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vect	-0.33	1.0			
alg	-0.23	-0.28	1.0		
anal	**0.00**	**-0.08**	-0.43	1.0	
stat	**-0.02**	**-0.02**	-0.36	-0.25	1.0
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The off-diagonal entries of the scaled inverse correlation matrix are the **negative of the conditional correlation coefficients** (i.e., the correlation coefficients after conditioning on the rest of the variables).

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Important: In general, uncorrelated variables are not independent. This is true however for the multivariate Gaussian distribution.

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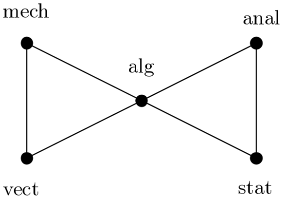

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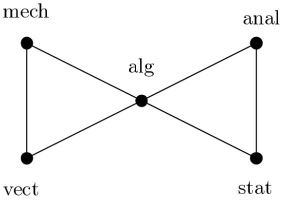

We put **no edge** between two variables iff they are conditionally independent (given the rest of the variables).

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- **Independent variables:** For two random vectors X, Y:

$$X \perp\!\!\!\perp Y \quad \Leftrightarrow \quad f_{X,Y}(x,y) = g(x)h(y) \qquad \forall x, y$$

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- **Conditionally independent variables:** Similarly, for three random vectors X, Y, Z:

$$X \perp\!\!\!\perp Y | Z \quad \Leftrightarrow \quad f_{X,Y,Z}(x,y,z) = g(x,z)h(y,z)$$

for all x, y and all z for which $f_Z(z) > 0$.

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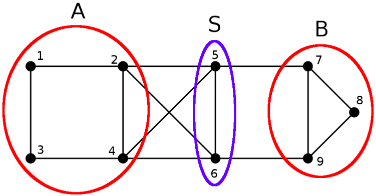
Theorem: (the separation theorem) Suppose the density of X is positive and continuous. Let $V = A \cup S \cup B$ be a partition of V such that S separates A from B. Then

$$X_A \perp\!\!\!\perp X_B \mid X_S.$$

Example: $X = (X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9)$:



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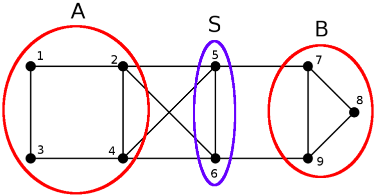

Then

$$(X_1, X_2, X_3, X_4) \perp\!\!\!\perp (X_7, X_8, X_9) | (X_5, X_6).$$

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2. The **local Markov property** if for every vertex $i \in V$,

$$X_i \perp\!\!\!\perp X_{V \setminus \text{cl}(i)} | X_{\text{ne}(i)},$$

where $\text{ne}(i) = \{j \in V : (i, j) \in E, j \neq i\}$ and $\text{cl(i)} = \{i\} \cup \text{ne}(i)$.

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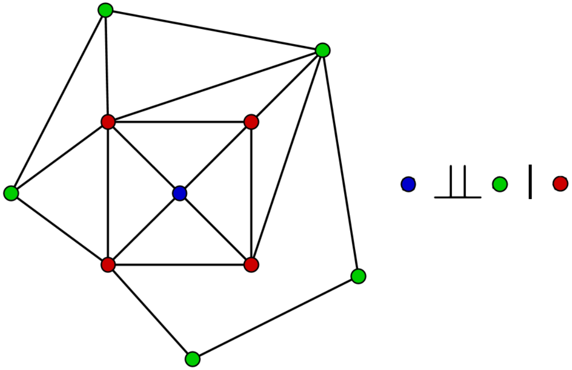
- When X has a positive and continuous density, by the separation theorem,
$$\text{pairwise} \Rightarrow \text{global}$$
and so all three properties are equivalent.

Example: the local Markov property

Illustration of the local Markov property:



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Theorem:(Hammersley–Clifford) Let X be a random vector with a positive and continuous density f. Then X satisfies the *pairwise Markov property* with respect to a graph G if and only if

$$f(x) = \prod_{C \in \mathcal{C}} \psi_C(x_C),$$

where $\mathcal{C}$ is the set of (maximal) cliques (complete subgraphs) of G.