MATH 829: Introduction to Data Mining and Analysis
Hidden Markov Models - Review of Markov chains

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May 9, 2016

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- When $X_n = j$, we say that the process is in state j at time n.

Stationarity and transition probabilities

- A Markov chain is **homogeneous** (or **stationary**) if for all $n \geq 0$ and all $i, j \in S$,

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- Conversely, every stochastic matrix is the transition matrix of some homogeneous discrete time Markov chain.

Examples

Example 1: (Two-state Markov chain)

$$S = \{0, 1\}, \quad p(0, 1) = a, \quad p(1, 0) = b, \quad a, b \in [0, 1]$$
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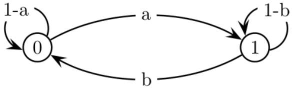

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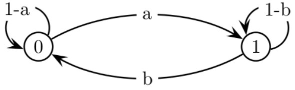

Interpretation: machine is either broken (0) or working (1) at start of n-th day.

Example 2: (Simple random walk) Let $\xi_1, \xi_2, \xi_3, \ldots$ be iid random variables such that $\forall i \geq 1$,

$$\xi_i = \begin{cases} +1 & P(\xi_i = +1) = p \\ 0 & P(\xi_i = 0) = r \\ -1 & P(\xi_i = -1) = q \end{cases},$$

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- The process is a **random walk**.

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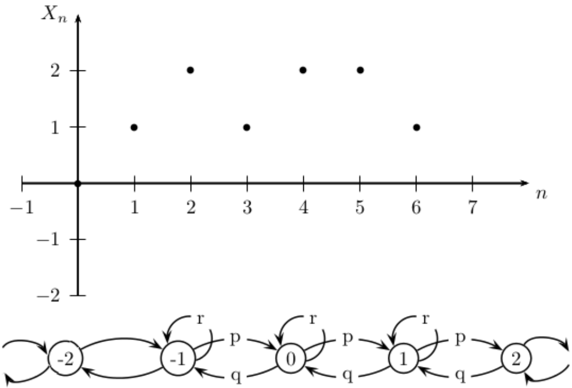

Review of Markov chains (cont.)

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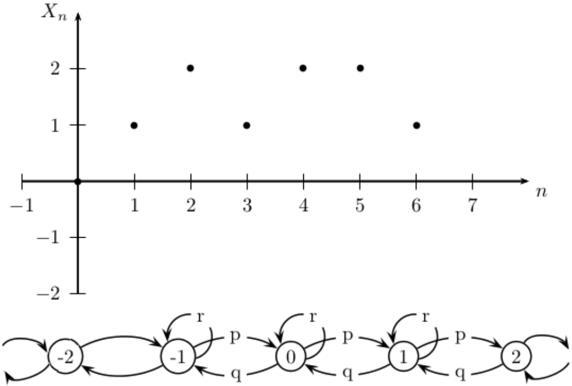

Exercise: What is P for that Markov chain?

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$$P^{(n)} := (p^n(i,j) : i,j \in S).$$

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Moral: Distributional computations for Markov Chains are just matrix multiplications.

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A Markov chain is said to be **irreducible** if its state space is a single communicating class.

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  A Markov chain is **aperiodic** if every state is aperiodic.

Limiting behavior

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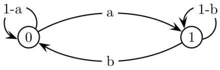

If $(a,b) \neq (0,0)$, we have (exercise):

$$P^n = \frac{1}{a+b}\begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{(1-a-b)^n}{a+b}\begin{pmatrix} a & -a \\ -b & b \end{pmatrix}.$$

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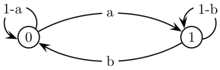

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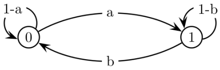

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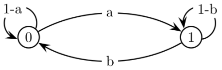
Thus, the chain has a **limiting distribution**.

Limiting behavior

Limiting behavior of Markov chains: What happens to $p^n(i,j)$ as $n \to \infty$?

Example: (The two-state Markov chain)



If $(a,b) \neq (0,0)$, we have (exercise):

$$P^n = \frac{1}{a+b}\begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{(1-a-b)^n}{a+b}\begin{pmatrix} a & -a \\ -b & b \end{pmatrix}.$$

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The limiting distribution is **independent of the initial state**.

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Recall: $\mu_{n+1} = \mu_n P$.

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$\pi(i)$ can be interpreted as the average proportion of time spent by the chain in state i.