MATH 829: Introduction to Data Mining and Analysis
A (very brief) introduction to Bayesian inference

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Frequentist statistics:

- Compute *point* estimates (e.g. maximum likelihood).
- Define probabilities as the long-run frequency of events .

Bayesian vs. Frequentist

Frequentist statistics:

- Compute *point* estimates (e.g. maximum likelihood).
- Define probabilities as the long-run frequency of events .

Bayesian statistics:

- Probabilities are a “state of knowledge” or a “state of belief”.
- Parameters have a probability distribution.
- Prior knowledge is updated in the light of new data.

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- Bayesian approach: we treat p as a *random variable*.
  1. Choose a *prior* distribution for p, say $P(p)$.
  2. Update the prior distribution using the data via *Bayes' theorem*:

$$P(p|data) = \frac{P(data|p)P(p)}{P(data)} \propto P(data|p)P(p).$$

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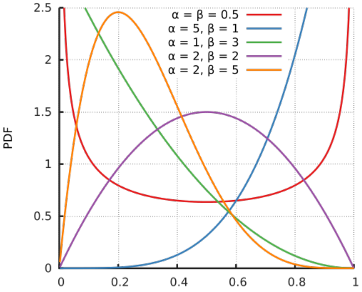
The beta distribution $\text{Beta}(\alpha, \beta)$:

$$P(p; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} \qquad (p \in (0,1)).$$



Source: Wikipedia.

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&= \binom{14}{10} p^{10}(1-p)^4 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} \\
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- Conclusion: $P(p|data) \sim \mathrm{Beta}(10+\alpha, 4+\beta)$.

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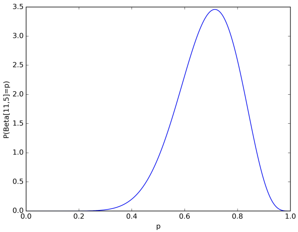

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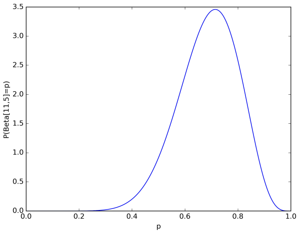

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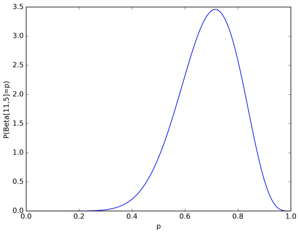

Our “knowledge” of p has now been updated using the observed data (or evidence).
Impportant advantage: Our estimate of p comes with its own **uncertainty**.

Bayesian analysis

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- Mimics the scientific method: formulate hypothesis, run experiment, update knowledge.
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Drawbacks:

- Not always obvious how to choose priors.
- Can be difficult to compute the posterior distribution.
- Can be computationally intensive to sample from the posterior distribution (when not available in closed form).

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- A prior with this property is said to be a *conjugating prior*.
- Conjugating priors are known for many common likelihood functions.

Likelihood	Conjugate prior
Binomial	Beta
Multinomial	Dirichlet
Poisson	Gamma
Normal	
μ unknown, σ^2 known	Normal
μ known, σ^2 unknown	Inverse Chi-Square
Multivariate Normal	
μ unknown, $\mathbf{V}$ known	Multivariate Normal
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MCMC methods

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- Idea:

1. Construct a **Markov chain** with the desired distribution as its **stationary distribution** π.
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- One generally then compute some *statistics* of the sample (e.g. mean, variance, mode, etc.).

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1. Draw $z \sim h(x)$ and $u \sim \mathrm{Uniform}[0, 1]$.
2. If $u < f(z)/(c \cdot g(z))$ accept the draw. Otherwise, discard z and repeat.

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Works well in some cases, but the rejection rate is often large and the resulting algorithm can be very inefficient.

Metropolis–Hastings algorithm

- Nicolas Metropolis (1915–1999) was an American physicist. He worked on the first nuclear reactors at the Los Alamos National Laboratory during the second world war. Introduced the algorithm in 1953 in the paper

  Equation of State Calculations by Fast Computing Machines

  with A. Rosenbluth, M. Rosenbluth, A. Teller, and E. Teller
- W. K. Hastings (Born 1930) is a Canadian statistician who extended the algorithm to the more general case in 1970.

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- The Metropolis–Hastings starts with an initial sample, and generate new samples using a *transition probability density* $q(x, y)$ (the *proposal distribution*).
- We assume
  - we can evaluate $f(x)$ at every x.
  - we can evaluate $q(x, y)$ at every x, y.
  - we can sample from the distribution $q(x, \cdot)$.

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$$R = \frac{f(y)q(y,x)}{f(x)q(x,y)}$$

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3. "Accept" the new sample y with probability $\min(1, R)$. If y is accepted, set $x_{i+1} := y$. Otherwise, $x_{i+1} = x_i$.

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Some difficulties:

- Choosing an efficient proposal distribution $q(x, y)$.
- How long should we wait for the Markov chain to converge to the desired distribution, i.e., how many samples should we burn?
- How long should we sample after convergence to make sure we sample in low probability regions?

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4. etc..