MATH 829: Introduction to Data Mining and Analysis
Least angle regression

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February 29, 2016

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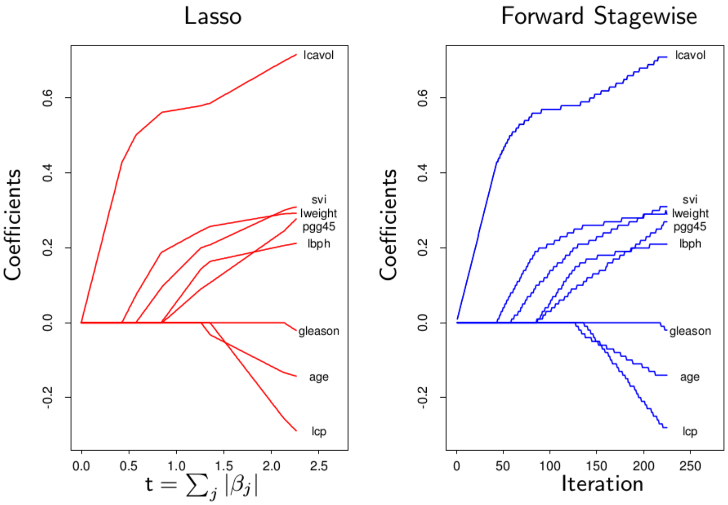
- However, the solution often looks similar to the lasso solution.
- Connection between the two methods?

Forward stagewise vs lasso

Example: Prostate cancer data (see ESL).



Efron et al., 2003

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Algorithm 3.2 *Least Angle Regression.*

1. Standardize the predictors to have mean zero and unit norm. Start with the residual $\mathbf{r} = \mathbf{y} - \bar{\mathbf{y}}$, $\beta_1, \beta_2, \ldots, \beta_p = 0$.
2. Find the predictor $\mathbf{x}_j$ most correlated with $\mathbf{r}$.
3. Move β_j from 0 towards its least-squares coefficient $\langle \mathbf{x}_j, \mathbf{r} \rangle$, until some other competitor $\mathbf{x}_k$ has as much correlation with the current residual as does $\mathbf{x}_j$.
4. Move β_j and β_k in the direction defined by their joint least squares coefficient of the current residual on $(\mathbf{x}_j, \mathbf{x}_k)$, until some other competitor $\mathbf{x}_l$ has as much correlation with the current residual.
5. Continue in this way until all p predictors have been entered. After $\min(N-1, p)$ steps, we arrive at the full least-squares solution.

ESL, Algorithm 3.2.

LARS (cont.)

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Then, at step k, we move the coefficients in the direction

$$\delta_k = (\mathbf{X}_{\mathcal{A}_k}^T \mathbf{X}_{\mathcal{A}_k})^{-1} \mathbf{X}_{\mathcal{A}_k}^T \mathbf{r}_k,$$

i.e., $\beta_{\mathcal{A}_k}(\alpha) = \beta_{\mathcal{A}_k} + \alpha \cdot \delta_k$.

- How does the correlation between the predictors and the residuals evolve?

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- Indeed, suppose each predictor in a linear regression problem has equal correlation (in absolute value) with the response.

$$\frac{1}{n} |\langle x_j, y \rangle| = \lambda \qquad j = 1, \ldots, p.$$

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  (Recall, we assume the predictors have been standardized.)
- Let $\hat{\beta}$ be the least-squares coefficients of y on X and let $u(\alpha) = \alpha X \hat{\beta}$ for $\alpha \in [0, 1]$.

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$$\begin{aligned}\left(\frac{1}{n}|\langle x_j, y - u(\alpha)\rangle|\right)_{j=1}^{p} &= \frac{1}{n}|X^T(y - u(\alpha))| \\ &= \frac{1}{n}|X^T(y - \alpha X(X^TX)^{-1}X^Ty)| \\ &= \frac{1}{n}|(1-\alpha)X^Ty| \\ &= (1-\alpha)\lambda \cdot \mathbf{1}_{p\times 1}.\end{aligned}$$

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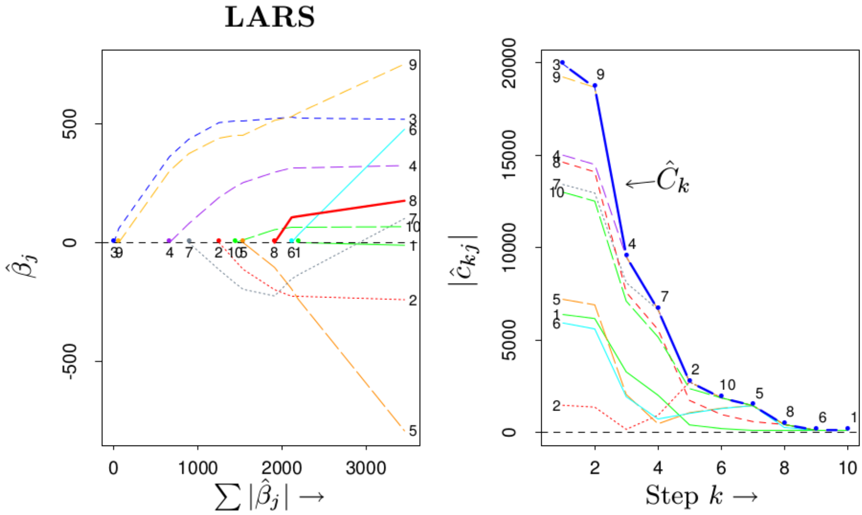
The new variable is then added to the model, and a new direction is computed.

Analysis of LARS (cont.)

Example: $\hat{C}_k$ = current maximal correlation.



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Why " least angle" regression?

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Problem: In general, given $v_1, \dots, v_k \in \mathbb{R}^n$, how do we find a vector that makes equal angles with $v_1, \dots, v_k$. When is this possible?

LARS and Lasso

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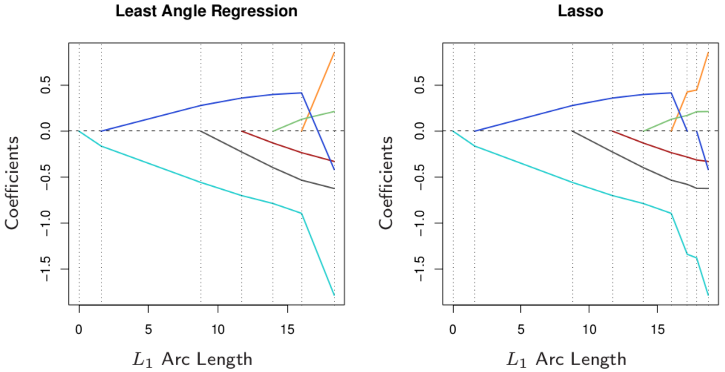

ESL, Figure 3.15.

On the above figure, the lasso coefficient profiles are almost identical to those of LARS in the left panel, and differ for the first time when the blue coefficient passes back through zero.

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Note: the theorem explains the *piecewise linear* nature of the lasso.

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- Recall: We proved before that the least-squares estimator is **consistent**, i.e.,

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  - Sparsity assumption (specified later).

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of $(Y, X_1, \ldots, X_p)$.

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We will provide a bound on $\mathrm{MSPE}(\tilde{\beta})$ when $\tilde{\beta}$ is the lasso solution.

Consistency of the Lasso (cont.)

Given $K > 0$, let $\tilde{\beta}^K = (\tilde{\beta}_1^K, \ldots, \tilde{\beta}_p^K)$ be the minimizer of

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Theorem: Under the previous assumptions and assuming

$$\sum_{j=1}^{p} |\beta_j^*| \leq K \text{ for some } K > 0 \text{ (sparsity assumption)},$$

we have

$$\mathrm{MSPE}(\tilde{\beta}^K) \leq 2KM\sigma\sqrt{\frac{2\log(2p)}{n}} + 8K^2M^2\sqrt{\frac{2\log(2p^2)}{n}}.$$

See Chatterjee, *Assumptionless consistency of the Lasso*, preprint, 2013.